ON GERBER-SHIU FUNCTIONS AND OPTIMAL DIVIDEND DISTRIBUTION FOR A LÉVY RISK-PROCESS IN THE PRESENCE OF A PENALTY FUNCTION

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Abstract. In this paper we consider an optimal dividend problem for an insurance company which risk process evolves as a spectrally negative Lévy process (in the absence of dividend payments). We assume that the management of the company controls timing and size of dividend payments. The objective is to maximize the sum of the expected cumulative discounted dividends received until the moment of ruin and a penalty payment at the moment of ruin which is an increasing function of the size of the shortfall at ruin; in addition, there may be a fixed cost for taking out dividends. We explicitly solve the corresponding optimal control problem. The solution rests on the characterization of the value-function as the smallest stochastic super-solution that we establish. We find also an explicit necessary and sufficient condition for optimality of a single dividend-band strategy, in terms of a particular Gerber-Shiu function.

Keywords: Stochastic optimal control; Lévy process; De Finetti problem; transaction costs; singular control; variational inequality; barrier policies, band policies; Gerber-Shiu function.

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1. Optimal control of Lévy risk models

The spectrally negative Lévy risk model. Recall the classical Cramér-Lundberg model

$$X_t - X_0 = \eta\, t - S_t, \qquad S_t = \sum_{k=1}^{N_t} C_k - \lambda\, m\, t, \tag{1.1}$$

which is used in collective risk theory (e.g. Gerber [21]) to describe the surplus $X = \{X_t, t \in \mathbb{R}_+\}$ of an insurance company. Here, C_k are i.i.d. positive random variables representing the claims made, $N = \{N_t, t \in \mathbb{R}_+\}$ is an independent Poisson process with intensity λ modelling the times at which the claims occur, and $p\, t$, with $p = \eta + \lambda m$, represents the premium income up to time t, with profit rate $\eta > 0$ and mean $m < \infty$ of C_1.

In later years, the model (1.1) has been generalized to the "perturbed model"

$$X_t - X_0 := \sigma B_t + \eta\, t - S_t, \tag{1.2}$$

where B_t denotes an independent standard Brownian motion, which models small scale fluctuations of the risk process.

Since the jumps of X are all negative, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$ and $t \in \mathbb{R}_+$, and is log-linear in t, defining thus a function $\psi(\theta)$ satisfying:

$$\mathbb{E}[\mathrm{e}^{\theta(X_t - X_0)}] = \mathrm{e}^{t\psi(\theta)},\ \psi(\theta) = \frac{\sigma^2}{2}\theta^2 + \eta\,\theta + \int_0^\infty (\mathrm{e}^{-\theta x} - 1 + \theta x)\nu(\mathrm{d}x), \tag{1.3}$$

where $\nu(\mathrm{d}x) = \lambda F_C(\mathrm{d}x)$, $x \in \mathbb{R}_+$, with F_C the distribution function of C_1, is the "Lévy measure" of the compound Poisson process S_t, and $\eta = \psi'(0)$ is the mean of $X_1 - X_0$.

The cumulant exponent $\psi(\theta)$ is well defined at least on the positive half-line, where it is strictly convex with the property that $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0), \infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. We shall denote the right-inverse function of ψ by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

An important generalization is to replace the process S in (1.2) by a general subordinator (a nondecreasing Lévy process, with Lévy measure $\nu(\mathrm{d}x), x \in \mathbb{R}_+$, which may have infinite mass). Under this model, the "small fluctuations" can arise either continuously, due to the Brownian motion, or due to the infinite mass of the Lévy measure.

Taking S to be a pure jump-martingale with i.i.d. increments and negative jumps with Lévy measure $\nu(\mathrm{d}x)$, one arrives thus to a general integrable spectrally negative Lévy process $X = \{X_t,\ t \in \mathbb{R}_+\}$ i.e. (see Bingham [15], Bertoin [12], Kyprianou [30]) a stochastic process that has stationary independent increments, no positive jumps and càdlàg paths with X_t integrable for any $t \in \mathbb{R}_+$, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is the natural filtration satisfying the usual conditions of right-continuity and completeness. To avoid degeneracies, we exclude the case that X has monotone paths. In case of zero Gaussian coefficient, X is called a pure-jump Lévy process. We denote

by $\{\mathbb{P}_x, x \in \mathbb{R}\}$ the family of probability measures that correspond to the shifts of X by a constant, that is, $\mathbb{P}_x[X_0 = x] = 1$.

An alternative characterization of spectrally negative Lévy processes is via the "q-harmonic homogeneous scale function" $W^{(q)}$, a non-decreasing function defined on the real line that is 0 on $(-\infty, 0)$, continuous on $\mathbb{R}_+$, with Laplace transform given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(y)\mathrm{d}y = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q). \tag{1.4}$$

Despite of the diversity of possible path behaviors displayed by spectrally negative Lévy processes, a wide variety of results may be elegantly expressed in a unifying manner via the homogeneous scale function $W^{(q)}$, bypassing thus "probabilistic complexity" via unified analytic methods. This paper further illustrates this aspect, by unveiling the way the scale function intervenes in a quite complex control problem.

De Finetti's dividend problem. Under the assumption that the increments of the surplus process have positive mean, the Lévy risk model has the unrealistic property that it converges to infinity with probability one.

In answer to this objection, De Finetti [17] introduced the risk process with dividends

$$U_t^\pi = X_t - D_t^\pi, \qquad t \geq 0, \tag{1.5}$$

where π is an "admissible" dividend control policy and D_t^π denotes the cumulative amount of dividends that has been transferred to a beneficiary up to time t, and where $U_{0-}^\pi = X_0 = x > 0$ is the initial capital.

Writing $\tau^\pi = \inf\{t \in \mathbb{R}_+ : U_t^\pi < 0\}$ for the time at which ruin occurs, the objective is to maximize the expected cumulative dividend payments until the time of ruin

$$v_*(x) := \sup_{\pi \in \Pi} \mathbb{E}_x \left[\int_0^{\tau^\pi} \mathrm{e}^{-qt} \mathrm{d}D_t^\pi \right],$$

where $\mathbb{E}_x[\cdot] = \mathbb{E}_x[\cdot | X_0 = x]$ and Π denotes the set of all admissible strategies, and where q is the discount rate.

Note that ruin may be either exogeneous or endogeneous (i.e. caused by a claim or by a dividend payment). A dividend strategy is admissible if ruin is always exogeneous, or more precisely, an admissible dividend strategy D^π is a right-continuous $\mathbf{F}$-adapted stochastic process that will satisfy that at any time preceding ruin, a dividend payment is smaller than the size of the available reserves:

$$\text{for any } t \leq \tau^\pi, \qquad \begin{cases} \Delta D_t^\pi := D_t^\pi - D_{t-}^\pi \leq \left(X_t - D_{t-}^\pi\right) \vee 0, & \text{and} \\ D_t^{\pi(c)} - D_u^{\pi(c)} \leq p(t-u) & \forall u \in [0,t), \text{ if } \nu_1 < \infty, \end{cases} \tag{1.6}$$

where $p := \eta + \nu_1$, $\nu_1 := \int_0^1 x\nu(\mathrm{d}x)$, and $D^{\pi(c)}$ denotes the continuous part of D^π.

The second line in Eqn. (1.6) states that, if the jump-part of X is of bounded variation, it is not admissible to pay dividends at a rate larger than the premium rate p at any time t that there are no reserves (i.e. $U_t^\pi = 0$), as this would lead to immediate ruin.

Single barrier policies. Recall first the simplest case when there are no transaction costs. One possible dividends distribution policy is the "barrier policy" π_b of transferring all surpluses above a given level b, which results in the optimal value:

$$v_b(x) := v_{\pi_b}(x) = \mathbb{E}_x\left[\int_0^{\sigma_b} \mathrm{e}^{-qt}\mathrm{d}D_t^b\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(b)}, \qquad x \in [0,b],$$

where $D^b = D^{\pi_b}$ is a local time-type strategy, given explicitly in terms of X by $D_{0-}^b = 0$ and

$$D_t^b = \sup_{s\le t}(X_s - b)^+, \qquad t \in \mathbb{R}_+,$$

with $x^+ = \max\{x, 0\}$. As this equation shows, a non-zero optimal barrier must be an inflection point of the scale function, if the latter is smooth.

Fixed transaction costs. It is interesting to consider also the effect of adding fixed transaction cost $K > 0$ that are not transferred to the beneficiaries when dividends are being paid. The objective of the beneficiaries becomes then to maximize $v_\pi(x)$:

$$v_*(x) = \sup_{\pi\in\Pi} v_\pi(x) \quad \text{where} \quad v_\pi(x) = \mathbb{E}_x\left[\int_0^{\tau^\pi} \mathrm{e}^{-qt}\mathrm{d}D_t^\pi - K\int_0^{\tau^\pi} \mathrm{e}^{-qt}\mathrm{d}N_t^\pi\right],$$

where N_t^π is the cardinality of the range $R(D^{-1})$ of the right-inverse D^{-1} of D^π, that is,

$$N_t^\pi = \#\{s \in [0,t] : s \in R(D^{-1})\}, \quad R(D^{-1}) = \{u \in \mathbb{R}_+ : D^{-1}(x) = u \text{ for some } x \in \mathbb{R}_+\}. \tag{1.7}$$

If the range $R(D^{-1})$ is discrete, N_t^π is equal to the number of times a dividend has been paid out by time t.

The introduction of a fixed transaction cost $K > 0$ has the usual effect of changing the optimal reflection boundaries b into strips $[b_-, b_+]$, so that when $U_t = b_+$, a dividend $b_+ - b_-$ is paid, and the process is diminished to the lower "entrance" point b_-.

The typical optimal dividend strategy consists then of "lump sum payments" [4], with π of the form $\pi = \{(J_k, T_k), k \in \mathbb{N}\}$, where $0 \le T_1 \le T_2 \le \ldots$ is an increasing sequence of $\mathbf{F}$-stopping times representing the times at which a dividend payment is made and $J_i \ge K$ is a sequence of positive $\mathcal{F}_{T_i}$-measurable random variables representing the sizes of the dividend payments. Then,

$$D_t^\pi = \sum_{k=1}^{N_t^\pi} J_k,$$

where $N_t^\pi = \#\{k : T_k \le t\}$ is the number of times that dividends have been paid by time t.

For single bands policies for example, the dividend distribution consists of the fixed amount $J_i = b_{i,+} - b_{i,-}$.

Optimality conditions for single band strategies. The interest in bands strategies was reawakened by Azcue and Muler [10], who considered the Cramér-Lundberg model via a viscosity approach, and produced the first example (with Gamma claims) in which a single constant band is not optimal. Let

$$b^* = \sup\{b > 0 : W^{(q)\prime}(b) \leq W^{(q)\prime}(x) \text{ for all } x\}, \tag{1.8}$$

denote the last global minimum of the scale derivative.

Avram et al. [8] showed that

$$(\Gamma v_{b^*} - q v_{b^*})(x) \leq 0, \qquad \text{for all } x > b^*, \tag{1.9}$$

where Γ denotes the infinitesimal generator of X, is a sufficient optimality condition for the single band strategy under a general spectrally negative Lévy model. In fact, the condition (1.8)–(1.9) is both necessary and sufficient.

Under the same model, Loeffen [33, 34] (with and without transaction cost) uncovered a **sufficient** single band optimality condition, namely that the last local minimum of the q-scale function is also a global minimum.

Kyprianou et al. [31] showed the optimality of a threshold policy if ν has a log-convex density. Loeffen and Renaud [35] provided a more general result by establishing optimality of the threshold policy in the presence of an affine penalty function with slope less than unity, if ν has a log-convex tail.

The optimality of barriers strategies was recently established by Albrecher and Thonhauser [2] in the presence of fixed interest rates as well.

Multiple barrier policies. However, single barrier strategies might not be optimal cf. Gerber [19, 20]. The optimal strategy may be a "multi-bands strategy", involving several "continuation bands" $[a_i, b_i), i = 0, 1, ...$ with upper reflecting boundaries b_i, separated by "dividend paying bands" $[b_i, a_{i+1}), i = 0, 1, ...$ of jumping to the next reflecting barrier below b_i, by paying all the excess as dividends (see also Hallin [26], who formulated a system of time dependent integro-differential equations associated to multi-bands policies). Gerber showed that for exponential claims (and with no constraints on the dividends rate), the optimal policy involved only one continuation band; however, constructing examples where more than one band was necessary remained an open problem for a long time.

Balancing dividends, ruin penalties and transaction costs. Several alternative objectives have been proposed recently, involving a penalty at ruin, based on a function of the severity of ruin [16, 22, 46], or on a continuous payoff until ruin [1].

The case where the insurance company is bailed out by the beneficiaries every time that there is a short fall in the reserves was investigated in [8], and in Kulenko and Schmidli [29].

In the current paper we investigate the influence of a general penalty and transaction costs on the optimal dividend policy. Assuming that the management of the company controls timing and size of dividend payments and is liable to pay a penalty that is a function of the shortfall at the moment of ruin, we solve the corresponding optimal control problem by constructing explicitly its solution. To show that the constructed function solves the stochastic optimal control problem, standard verification arguments that rely on the application of Itô's lemma can in general not be employed, due to a lack of smoothness of the value function. In particular, it will follow from the form of the value-function and from results concerning the smoothness of scale functions (Kyprianou *et al.* [31], Lambert [32]) that, in general, the value-function is continuous but not C^1 on $\mathbb{R}_{++} := (0, \infty)$ if X has bounded variation, and is C^1 but not C^2 on $\mathbb{R}_{++}$, if X has unbounded variation. The approach followed in this paper is probabilistic in nature and rests on a dual representation of the value function as the point-wise minimum of stochastic super-solutions (Thm. 4.2(i)), and on a comparison and local-verification result (Thm. 4.2(ii)), which is a consequence of this representation.

A key point is an explicit formula in Eqn. (2.24) below of a"continuous q-harmonic/Gerber-Shiu function" $F_w(x)$ associated to a given penalty $w(x), x \in \mathbb{R}_-$, in terms of simpler scale functions. Informally, $F_w(x)$ is the "continuous nonhomogeneous solution" of the Dirichlet problem on $\mathbb{R}_{++}$ with boundary condition $w(x), x \in \mathbb{R}_-$. More precisely, $F_w(x)$ is defined by subtracting a multiple of the homogeneous scale function $W^{(q)}(x)$ out of the solutions of either the two-sided, or the reflected exit problem, as defined in Section 2, such that the remaining part is *continuous* on $\mathbb{R}$ if w is continuous, and *continuously differentiable* on $\mathbb{R}$ if w is continuously differentiable on $\mathbb{R}_-$ and X has unbounded variation – see Definitions 2.3 and 2.8 in Section 2.2.

For exponential penalties $w(x) = \mathrm{e}^{xv}$, the Gerber-Shiu function has a simple formula (2.16), which may be used also as a generating function for the expected payoffs associated to polynomial penalties $x^k, k = 0, 1, ...$

The function $F_w(x)$ provides an explicit expression for the solution of the key auxiliary problem of finding the value function of a single dividend-band strategy and is equal to the value-function of an embedded optimal stopping problem – see Proposition 5.1 and Theorem 5.14, leading ultimately to the optimal band levels – see Section 5.4.

We establish an explicit *necessary* and *sufficient* criterion for optimality of single dividend barrier policies in the presence of fixed transaction cost and general penalty, which in particular includes the case of zero penalty ($w \equiv 0$) — see Theorem 5.4 in Section 5.1.

Contents. The remainder of the paper is organized as follows. Section 2 is concerned with two stochastic boundary value problems associated to the value of dividend payments in the presence of a penalty. In Section 3 the dividend-penalty problem is phrased and its optimal solution is presented, and Section 4 is devoted to the martingale approach. In Section 5 the value function is constructed,

and some examples are analyzed in detail in Section 6. A number of the proofs are presented in the Appendix.

2. Two stochastic boundary value problems

The problem under consideration in this section is the identification of the solution of two exit problems in terms of the q-scale function $W^{(q)}$. It is known that, on $\mathbb{R}_+$, $W^{(q)}$ is non-decreasing and everywhere right- and left-differentiable. Furthermore, if the Gaussian coefficient σ is positive, then $W^{(q)}|_{\mathbb{R}_+}$ is C^2 (see [31]), and $W^{(q)\prime}(0+) = \frac{2}{\sigma^2}$. Throughout the paper, we denote by $f'(x)$ the right-derivative at x of a function f.

Problem A. Given $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+ \cup \{+\infty\}$, $a < b$, let T_b^+, T_a^- be the first entrance times of X into the sets (b, ∞) and $(-\infty, a)$,

$$T_b^+ = \inf\{t \in \mathbb{R}_+ : X_t > b\}, \qquad T_a^- = \inf\{t \in \mathbb{R}_+ : X_t < a\},$$

with $\inf \emptyset = +\infty$, and let $T_{a,b} = T_a^- \wedge T_b^+$ denote the "two-sided" exit time from the interval $[a, b]$. The quantity of interest is the function $\mathcal{V}_w^{a,b} : (a, b) \to \mathbb{R}$ given by

$$\mathcal{V}_w^{a,b}(x) = \mathbb{E}_x \left[\exp\{-qT_{a,b}\}\, w\left(X_{T_a^-}\right) \mathbf{1}_{\{T_a^- < T_b^+\}}\right] + \delta \mathbb{E}_x \left[\exp\{-qT_{a,b}\} \mathbf{1}_{\{T_a^- > T_b^+\}}\right], \tag{2.1}$$

for $q \in \mathbb{R}_+$, $\delta \in \mathbb{R}$ and any given Borel-measurable function $w : (-\infty, a] \to \mathbb{R}$ (the "pay-off") satisfying the integrability condition

$$\int_{(b,\infty)} |w(x - y)|\nu(\mathrm{d}y) < \infty \qquad \text{for all } x \in [a, b]. \tag{2.2}$$

To $\mathcal{V}_w^{a,b}$ is associated the following stochastic boundary value problem (SBV) on (a, b): find a Borel-measurable function $f : (-\infty, b] \to \mathbb{R}$ such that

$$\mathrm{e}^{-q(t\wedge T_{a,b})} f\left(X_{t\wedge T_{a,b}}\right) \text{ is a UI } \mathbb{P}_x\text{-martingale, for any } x \in (a, b), \tag{2.3}$$

$$f(b) = \delta, \qquad \text{if } b < \infty, \tag{2.4}$$

$$f|_{(-\infty,a)} = w|_{(-\infty,a)}, \tag{2.5}$$

$$f(a) = w(a), \qquad \text{if } \nu_1 = \infty \text{ or } \sigma > 0, \tag{2.6}$$

where $\nu_1 = \int_0^1 x\nu(\mathrm{d}x))$ and where, if $b = \infty$, the boundary condition (2.4) is replaced by the condition $\limsup_{x\to\infty} f(x) < \infty$.

Problem B. For any $a, b \in \mathbb{R}$ with $a < b$, let

$$\tau_a = \inf\{t \in \mathbb{R}_+ : Y_t^b < a\}$$

be the first-passage time into (a, ∞) of the process $Y^b = \{Y_t^b, t \in \mathbb{R}_+\}$ that is equal to the process X reflected at the level b,

$$Y_t^b = X_t - \overline{X}_t^b \qquad \text{with} \qquad \overline{X}_t^b = \sup_{s\le t} (X_t - b) \vee 0.$$

Fixing $\beta \in \mathbb{R}$, and $w : (-\infty, a] \to \mathbb{R}$ satisfying (2.2) the function of interest in this case is $\mathcal{U}_w^{a,b} : (a,b) \to \mathbb{R}$ given by

$$\mathcal{U}_w^{a,b}(x) = \mathbb{E}_x\left[\exp\{-q\tau_a\} w\left(Y^b_{\tau_a}\right)\right] + \beta\,\mathbb{E}_x\left[\int_0^{\tau_a} \mathrm{e}^{-qs}\mathrm{d}\overline{X}^b_s\right]. \tag{2.7}$$

The corresponding SBV problem is to find a Borel-measurable function $f : (-\infty, b] \to \mathbb{R}$ such that

$$\mathrm{e}^{-q(t\wedge\tau_a)} f\left(Y^b_{t\wedge\tau_a}\right) + \beta \int_0^{t\wedge\tau_a} \mathrm{e}^{-qs}\mathrm{d}\overline{X}^b_s \text{ is a UI } \mathbb{P}_x\text{-martingale, } x \in (a,b), \tag{2.8}$$

$$f|_{(-\infty,a)} = w|_{(-\infty,a)}, \tag{2.9}$$

$$f(a) = w(a), \qquad \text{if } \nu_1 = \infty \text{ or } \sigma > 0. \tag{2.10}$$

We will show in Thm. 2.12 below that Problems A and B admit unique solutions if w is sufficiently regular. The boundary condition at a is imposed only if X has unbounded variation, in which case a is regular for $(-\infty, a)$ for X and Y^b. We will drop the super-scripts a, b if $a = 0$ and/or $b = \infty$, writing $\mathcal{V}_w := \mathcal{V}_w^{0,\infty}$ and $\mathcal{U}_w^{0,b} := \mathcal{U}_w^b$.

Certain particular cases of Problems A and B have been extensively studied in actuarial science. For example, $\mathcal{U}_w^b$ with $\beta = 1$ and $w \equiv 0$ is equal to the present value of the cumulative dividend payments under a barrier strategy at level b. The following particular cases will be needed in the sequel.

Example 2.1. In the case that b is finite, $w \equiv 0$, the unique solutions of Problems A and B are given by

$$\mathcal{V}_w^{a,b}(x) = W^{(q)}(x-a)\frac{\delta}{W^{(q)}(b-a)}, \qquad \mathcal{U}_w^{a,b}(x) = W^{(q)}(x-a)\frac{\beta}{W^{(q)\prime}(b-a)}, \tag{2.11}$$

for $x \in (a, b]$ ([13, Thm. 1] and [8, Prop. 1]).

Example 2.2. In case $w \equiv 1$, the unique solutions of Problem A with $b < \infty$ and $\delta = 0$ and of Problem B with $\beta = 0$ (unit pay-off without dividends) are given by

$$\mathcal{V}_w^{a,b}(x) = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}, \tag{2.12}$$

$$\mathcal{U}_w^{a,b}(x) = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{qW^{(q)}(b-a)}{W^{(q)\prime}(b-a)}, \tag{2.13}$$

for $x \in [a, b]$, where the function $Z^{(q)}$ is given by $Z^{(q)}(x) = 1 + q\overline{W}^{(q)}(x)$, with $\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(y)\mathrm{d}y$, the anti-derivative of $W^{(q)}$ ([13, Thm. 1] and [7, Thm. 1]).

Definition 2.3. Let $a < b < \infty$, $\delta, \beta \in \mathbb{R}$ and *pay-off* $w : (-\infty, a] \to \mathbb{R}$ be given. We will call $F : \mathbb{R} \to \mathbb{R}$ a *Gerber-Shiu function for payoff* w if $F|_{\mathbb{R}_{++}}$ is right-differentiable and the following hold:

(i) Problem A admits a unique solution given by

$$\mathcal{V}_w^{a,b}(x) = F(x-a) + W^{(q)}(x-a)\frac{\delta - F(b-a)}{W^{(q)}(b-a)}, \qquad x \in (a,b). \tag{2.14}$$

(ii) Problem B admits a unique solution given by

$$\mathcal{U}_w^{a,b}(x) \quad = \quad F(x-a) + W^{(q)}(x-a)\frac{\beta - F'(b-a)}{W^{(q)\prime}(b-a)}, \qquad x \in (a,b). \tag{2.15}$$

Of course, such a function F is not unique. In the next sections, we construct special Gerber-Shiu functions that are **continuous** on $\mathbb{R}$ for continuous payoffs w and **continuously differentiable** on $\mathbb{R}$ if X has unbounded variation and w is continuously differentiable, starting with the case of exponential and polynomial payoffs (note that neither $\mathcal{V}_w^{a,b}$, $\mathcal{U}_w^{a,b}$, nor $W^{(q)}$ are continuous or continuously differentiable on $\mathbb{R}$ in general).

2.1. **Exponential and polynomial boundary conditions.** In the case that w is exponential or polynomial the solution of the stochastic boundary value problems can be expressed in terms of the following family of functions:

Definition 2.4. For $q \in \mathbb{R}_+$, $v \in \mathbb{R}_+$, the function $Z^{(q,v)} : \mathbb{R} \to \mathbb{R}$ is defined by $Z^{(q,v)}(x) = \mathrm{e}^{vx}$ for $x \le 0$ and, for $x > 0$, by

$$Z^{(q,v)}(x) = \mathrm{e}^{vx} + (q - \psi(v)) \int_0^x \mathrm{e}^{v(x-y)} W^{(q)}(y)\mathrm{d}y. \tag{2.16}$$

With n the largest integer such that $\int_{-\infty}^{-1} |x|^n \nu(\mathrm{d}x) < \infty$, the related family of functions $Z_k : \mathbb{R} \to \mathbb{R}$, $k = 0, \dots n$, is defined by

$$Z_k(x) = \left.\frac{\partial^k}{\partial v^k}\right|_{v=0+} Z^{(q,v)}(x). \tag{2.17}$$

Note that, for any $q, v \in \mathbb{R}_+$, $Z^{(q,v)}|_{\mathbb{R}_+}$ is C^1, as a consequence of the continuity of $W^{(q)}|_{\mathbb{R}_+}$.

The composition of a function f with a translation θ_a by $a \in \mathbb{R}$ will be denoted by

$$_af := f \circ \theta_a := f(\cdot + a). \tag{2.18}$$

Let $e_v(x) := \mathrm{e}^{vx}\mathbf{1}_{\mathbb{R}_-}(x)$ denote an exponential pay-off, and $e_{v,a} :=_{-a} e_v$ the translated version. The solutions of Problems A and B with $\delta = \beta = 0$ and pay-off $e_{v,a}$ are given as follows:

Proposition 2.5. *For $q \in \mathbb{R}_+$ and $v \in \mathbb{R}_+$, $Z^{(q,v)}$ is a Gerber-Shiu function with payoff $e_{v,a}$. In particular, the following hold true:*

$$\mathcal{V}_{e_{v,a}}^{a,b}(x) = \mathbb{E}_x\left[\mathrm{e}^{-qT_{a,b} + v(X_{T_{a,b}} - a)}\mathbf{1}_{\{T_a^- < T_b^+\}}\right] = Z^{(q,v)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q,v)}(b-a)}{W^{(q)}(b-a)}, \tag{2.19}$$

$$\mathcal{U}_{e_{v,a}}^{a,b}(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a + v(Y_{\tau_a}^a - a)}\right] = Z^{(q,v)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q,v)\prime}(b-a)}{W^{(q)\prime}(b-a)}. \tag{2.20}$$

For use in the sequel we record the special case of the kth moment of the overshoot

$$m_k(x) := \mathbb{E}_x\left[\mathrm{e}^{-qT_{a,b}}(X_{T_{a,b}} - a)^k \mathbf{1}_{\{T_a^- < T_b^+\}}\right], \qquad \tilde{m}_k(x) := \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a}(Y_{\tau_a}^a - a)^k\right], \tag{2.21}$$

with $\mathbf{1}_A$ denoting the indicator of a set A, which can be derived as a direct consequence of Prop. 2.5. If $E[|X_1|^k] < \infty$, then $\psi^{(r)}(0)$ and $m_r(x)$ are finite for $r = 1, \ldots, k$, and it follows that $m_k(x)$ and $\tilde{m}_k(x)$ are equal to the kth derivative of (2.19) and (2.20) with respect to v at $v = 0$. This implies the following form of $m_k(x)$ and $\tilde{m}_k(x)$:

Corollary 2.6. *Let $k \in \mathbb{N}$. Suppose that $\int_{-\infty}^{-1} |x|^k \nu(\mathrm{d}y) < \infty$. Then, for $x \in [a, b]$, $m_k(x)$ and $\tilde{m}_k(x)$ are finite, and are given by*

$$m_k(x) = Z_k(x-a) - W^{(q)}(x-a)\frac{Z_k(b-a)}{W^{(q)}(b-a)}, \quad \tilde{m}_k(x) = Z_k(x-a) - W^{(q)}(x-a)\frac{Z_k'(b-a)}{W^{(q)\prime}(b-a)}.$$

In particular, Z_k is a Gerber-Shiu function with payoff $w(x) = (x-a)^k$.

The proofs of Prop. 2.5 and Cor. 2.6 are given in Appendix A.

2.2. **General boundary functions.** We shall restrict ourselves to the following class of payoffs w.

Definition 2.7. We denote by $\mathcal{R}$ the set of Borel-measurable functions $w : \mathbb{R}_- \to \mathbb{R}$ that are continuous at 0, admit a finite left-derivative $w'(0-)$ at 0, and satisfy the following integrability conditions:

$$\text{(2.22)} \quad \text{(i)} \ \ w_\nu(y) < \infty \ \forall y \in \mathbb{R}_{++} \qquad \text{and} \qquad \text{(ii)} \ \ \int_0^\infty \int_y^\infty \mathrm{e}^{-\Phi(q)y} |w(y-z) - w(0)| \nu(\mathrm{d}z)\mathrm{d}y < \infty,$$

where the function $w_\nu : \mathbb{R}_{++} \to \mathbb{R}$ is defined by

$$\text{(2.23)} \qquad w_\nu(y) := \int_{(y,\infty)} \{w(y-z) - w(0)\}\, \nu(\mathrm{d}z) \qquad y \in \mathbb{R}_{++}.$$

To each payoff $w \in \mathbb{R}$ we associate a scale function F_w:

Definition 2.8. Let $q \in \mathbb{R}_+$ and $w \in \mathcal{R}$. The function $F_w : \mathbb{R} \to \mathbb{R}$ is given by $F_w(x) = w(x)$ for $x < 0$, and is continuous on $\mathbb{R}_+$ with Laplace transform

$$\text{(2.24)} \qquad \int_0^\infty \mathrm{e}^{-\theta x} F_w(x)\mathrm{d}x = (\psi(\theta) - q)^{-1} \left[\frac{\sigma^2}{2}[w'(0-)] + \frac{\psi(\theta)}{\theta} w(0) - w_\nu^*(\theta)\right], \qquad \theta > \Phi(q),$$

where w_ν^* denotes the Laplace transform of w_ν.

Remark 2.9. (i) In the case of exponential penalty, $w = e_v$, the function F_w reduces to $Z^{(q,v)}$. Indeed, the Laplace transforms $F_{e_v}^*$ and $(Z^{(q,v)})^*$ of $F_{e_v}|_{\mathbb{R}_+}$ and $Z^{(q,v)}|_{\mathbb{R}_+}$ are both equal to:

$$F_{e_v}^*(\theta) = (Z^{(q,v)})^*(\theta) = (\psi(\theta) - q)^{-1} \frac{\psi(\theta) - \psi(v)}{\theta - v}.$$

(ii) Properties of F_w that will be used in the sequel are listed in Appendix A.

The classical Gerber-Shiu function $\mathcal{V}_w^{0,\infty}(x)$ corresponding to penalty w can be explicitly expressed in terms of F_w, as follows:

Proposition 2.10 (Classical Gerber-Shiu function). *Let $w \in \mathcal{R}$. For any $x \in \mathbb{R}$ it holds that*

$$\mathcal{V}_w^{0,\infty}(x) = \mathbb{E}_x \left[\exp\left\{-qT_0^-\right\} w(X_{T_0^-})\mathbf{1}_{\{T_0^-<\infty\}}\right] = F_w(x) - W^{(q)}(x)\kappa_w, \tag{2.25}$$

where

$$\kappa_w := \left[\frac{\sigma^2}{2}w'(0-) + \frac{q}{\Phi(q)}w(0) - w_\nu^*(\Phi(q))\right]. \tag{2.26}$$

In particular, the following martingale property holds true:

$$\left(\mathrm{e}^{-q(t\wedge T_a^-)}F_w(X_{t\wedge T_a^-} - a), t \in \mathbb{R}_+\right) \qquad \text{is a } \mathbb{P}_x\text{-martingale, for any } x, a \in \mathbb{R}. \tag{2.27}$$

Remark 2.11. An equivalent representation for $\mathcal{V}_w^{0,\infty}$ in terms of $W^{(q)}$ was found in Biffis & Kyprianou [14].

The solutions of the stochastic boundary value problems can hence be expressed in terms of the functions $W^{(q)}$ and F_w as follows:

Theorem 2.12. *Let $w \in \mathcal{R}$. The function $F_{{}_aw}$ is a Gerber-Shiu function for the payoff w, where ${}_aw$ denotes the translation of w defined in Eqn.* (2.18).

Proof of Theorem 2.12: An application of the compensation formula yields the following representation of $\mathcal{U}_w^{a,b}(x)$:

$$\mathcal{U}_w^{a,b}(x) - w(0)\mathcal{U}_{e0,a}^{a,b}(x) = \int_a^b \int_y^\infty (w(y-z) - w(0))\nu(\mathrm{d}z)R_{a,b}^q(x,\mathrm{d}y),$$

where $\mathcal{U}_{e0,a}^{a,b}$ is given in (2.20), and q-resolvent measure $R_{a,b}^q(x,\mathrm{d}y)$ of Y^b killed upon entering $(-\infty, a)$ which is given by ([38, Thm. 1])

$$R_{a,b}^q(x,\mathrm{d}y) = \frac{W^{(q)}(x-a)}{W^{(q)\prime}(b-a)}W^{(q)}(b-\mathrm{d}y) - W^{(q)}(x-y)\mathrm{d}y, \qquad x, y \in [a,b]. \tag{2.28}$$

Combining these expressions with Lem. A.2(iii) and taking note of the fact that the term $\frac{\sigma^2}{2}{}_aw'(0-)W^{(q)}(x)$ cancels yields that Eqn. (2.15) holds with $F = F_{{}_aw}$.

Next we turn to the uniqueness. In view of the definition (2.7) of $\mathcal{U}_w$, the strong Markov property of Y^b and the integrability condition (2.22), the process

$$\mathrm{e}^{-q(t\wedge\tau_a)}\mathcal{U}_w(Y_{t\wedge\tau_a}^b) + \beta\int_0^{t\wedge\tau_a} \mathrm{e}^{-qs}\mathrm{d}\overline{X}_s^b = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a}w(Y_{\tau_a}^b) + \beta\int_0^{\tau_a}\mathrm{e}^{-qs}\mathrm{d}\overline{X}_s^b \middle| \mathcal{F}_t\right], \tag{2.29}$$

is a UI $\mathbb{P}_x$-martingale, for $x \in [a,b]$, and is hence a solution of Problem B. If $\mathcal{U}^1, \mathcal{U}^2$ are two solutions of Problem B then $\mathcal{U}_0 := \mathcal{U}^1 - \mathcal{U}^2$ is a solution of (2.8) – (2.9) with $w = \beta = 0$. Hence,

$$\mathcal{U}_0(x) = \lim_{t\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-q(t\wedge\tau_a)}\mathcal{U}_0(Y_{t\wedge\tau_a}^b)\right] = 0, \tag{2.30}$$

where we used that $\mathrm{e}^{-q(t\wedge\tau_a)}\mathcal{U}_0(Y^b_{t\wedge\tau_a})$ is uniformly integrable. A similar argument yields the uniqueness of a solution to Problem A. As for the existence, an application of the strong Markov property yields the following:

$$\mathbb{E}_x\left[\exp\{-qT_{a,b}\}\, w\left(X_{T_{a,b}}\right)\right] = \mathcal{V}^{a,\infty}_w(x) - \mathbb{E}_x\left[\exp\{-qT_{a,b}\}\,\mathbf{1}_{\{T^+_b<T^-_a\}}\right](\delta+\mathcal{V}^{a,\infty}_w(b)). \tag{2.31}$$

Combining Eqn. (2.1) with Eqn. (2.31), and inserting (2.11) and (2.25), yields the identity (2.14) with $F = F_{{}_a w}$ (using that the term $\frac{\sigma^2}{2}{}_a w'(0-)W^{(q)}(x)$ cancels). □

3. The dividend-penalty control problem

The process X represents the cash-reserves of the company in the absence of dividend payments. We will restrict ourselves to the case of positive net income (or infinitesimal drift), $\eta := \mathbb{E}[X_1] > 0$. The level of the reserves $\{U_t, t\in\mathbb{R}_+\}$ when the dividend payments has been taken into account is then given by

$$U_t = X_t - D_t,$$

where $D = \{D_t, t\in\mathbb{R}_+\}$ is a dividend process, a non-decreasing right-continuous $\mathbf{F}$-adapted process with $D_{0-} = 0$. Here D_t represents the cumulative amount of dividends that has been paid until time t. The beneficiaries control the timing and size of dividend payments made by the company, and are liable to pay at the moment τ^π of ruin the penalty $-w(U^\pi_{\tau^\pi})$, which may be used to cover (part of) the claim that led to insolvency, where w is a penalty. The beneficiaries seek to pay out dividends according to an admissible policy that maximises the sum of the expected discounted cumulative dividends and the expected penalty payment.

Definition 3.1. A *penalty* $w:\mathbb{R}_-\to\mathbb{R}_-$ is an increasing function that is right-continuous on $(-\infty,0)$, left-continuous at 0 and admits a finite left-derivative $w'(0-)$, and satisfies the integrability condition

$$\int_1^\infty |w(-z)|\nu(\mathrm{d}z) < \infty. \tag{3.1}$$

The collection of penalties is denoted by $\mathcal{P}$. It is straightforward to verify that $\mathcal{P}\subset\mathcal{R}$, the class given in Definition 2.7.

The present value of the penalty payment discounted at rate $q>0$, considered as function of the level of reserves, is called the "Gerber-Shiu penalty function" associated to the penalty w, and is given by

$$\mathcal{W}^\pi_w(x) := \mathbb{E}_x\left[\mathrm{e}^{-q\tau^\pi} w\left(U^\pi_{\tau^\pi}\right)\right], \qquad x\in\mathbb{R}_+,$$

where $\mathbb{E}_x[\cdot]$ denotes the expectation under $\mathbb{P}_x$. Under condition (3.1), it holds that, for any level of initial capital $x\in\mathbb{R}_+$, $\mathcal{W}^\pi_w(x)$ is bounded uniformly over $\pi\in\Pi$ (see Lemma 4.9).

The objective of the beneficiaries of the insurance company is phrased in terms of the following stochastic optimal control problem:

$$v_*(x) = \sup_{\pi\in\Pi} v_\pi(x), \qquad v_\pi(x) := \mathcal{W}^\pi_w(x) + \mathbb{E}_x\left[\int_0^{\tau^\pi} \mathrm{e}^{-qt}\mu_K(\mathrm{d}t)\right], \qquad x\in\mathbb{R}_+, \tag{3.2}$$

where Π denotes the set of admissible dividend policies π and μ_K is the (possibly signed) random measure on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ defined by

$$\mu^\pi_K([0,t]) = D^\pi_t - KN^\pi_t, \tag{3.3}$$

with N^π_t equal to the counting process defined in Eqn. (1.7). The solution to the stochastic control problem (3.2) consists of a pair (w,π^*) of a function $w:\mathbb{R}_+\to\mathbb{R}$ and a policy $\pi^*\in\Pi$ such that $v_*(x) = w(x) = v_{\pi^*}(x)$ for all $x\in\mathbb{R}_+$.

A flexible class of dividend strategies are the so-called multi dividend-bands strategies, that are specified as follows:

Definition 3.2. The *multi dividend-bands strategy* $\pi_{\underline{a},\underline{b}}$, associated to sequences $\underline{a}=(a_n)_n$, $\underline{b}^-=(b^-_n)_n$, $\underline{b}^+=(b^+_n)_n$ with $a_n, b^-_n, b^+_n\in[0,\infty]$ satisfying the intertwining conditions

$$a_0 = 0\le b^+_1 < a_1 \le b^+_2 < \ldots < a_{n-1}\le b^+_n < \ldots, \qquad b^-_n \le b^+_n,$$

is described as follows:

(i) When $U^{\underline{a},\underline{b}} := U^{\pi_{\underline{a},\underline{b}}} = y\in(b^+_n, a_{n+1})$, make a lump-sum payment $y-b^-_n$;
(ii) When $U^{\underline{a},\underline{b}} = b^+_n$ make a lump-sum payment $b^+_n - b^-_n$, if $K>0$, and pay the minimal amount to keep $U^{\underline{a},\underline{b}}$ below $b^-_n = b^+_n$ if $K=0$;
(iii) While $U^{\underline{a},\underline{b}}\in[a_n,b^+_n)$ do not pay any dividends.

The strategy $\pi^{\underline{a},\underline{b}}$ is called a *single dividend-band strategy* if $b^+_1<\infty = a_2$.

In the case of zero transaction cost a multi dividend-bands strategy $\pi_{\underline{a},\underline{b}}$ consists in paying out "the minimal amount to keep $U^{\underline{a},\underline{b}}_t$ below the boundary $b(t)$", where

$$b(t) = b^+_{\rho(t)} \qquad \text{with } \rho(t) = \min\{i\ge 1: X_t < a_i\};$$

the corresponding dividend process $D^{\pi_{\underline{a},\underline{b}}} = \{D^{\pi_{\underline{a},\underline{b}}}_t, t\in\mathbb{R}_+\}$ is explicitly given by

$$D^{\pi_{\underline{a},\underline{b}}}_t := D^{\underline{a},\underline{b}}_t = \sup_{s\le t}\left(X_s - b(s)\right)^+, \qquad t>0,$$

and is equal to a local time of $U^{\underline{a},\underline{b}}$ at the boundary $b=(b(t))_{t\in\mathbb{R}_+}$. Note that in this case the process X is reflected at the levels b^+_n: the dividend strategy is the minimal moderation of X to ensure that the moderated process does not cross the boundary $b(t)$. In the case of a positive fixed transaction cost K the "reflection boundaries" b^+_n widen to strips $[b^-_n, b^+_n]$ and the "local time" type payments are replaced by lump-sum payments $b^+_n - b^-_n$ where b^-_n may lie below a_{n-1} (see Figure 3).

The solution of the stochastic control problem is given as follows:

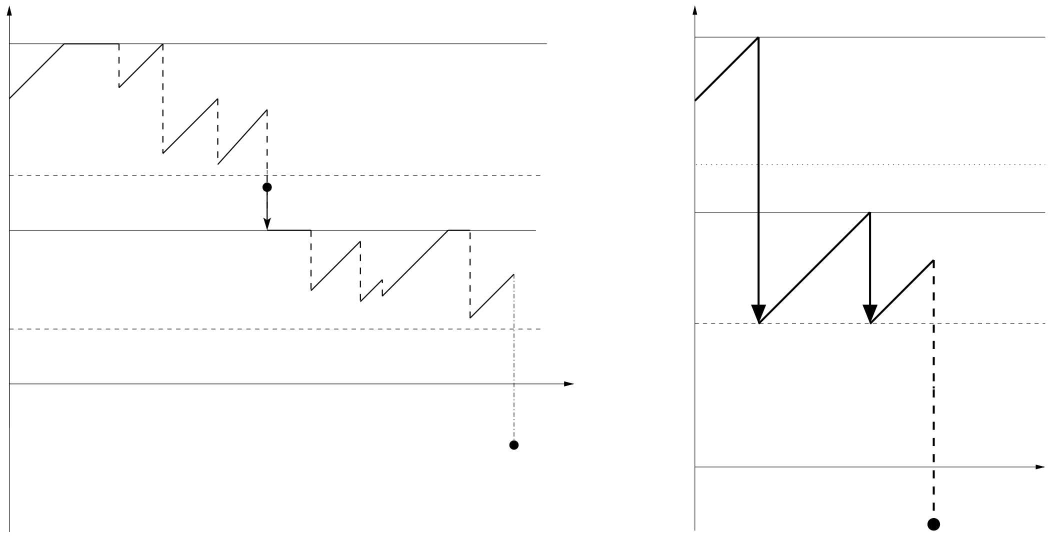

FIGURE 1. Illustrated in the figure on the left is a path of the risk process U^π in the absence of transaction cost ($K = 0$) for a three-bands strategy with the lowest level b_1^+ equal to zero. The figure on the right pictures a path of the risk process U^π in the case $K > 0$ and π is a two-bands strategy with $b_2^- = b_1^-$. The vertical dashed stretches represent the claims, while lump sum dividend payments are indicated by arrows. At the moment τ of ruin a penalty payment $w(U_\tau)$ is required that is a function of the shortfall U_τ

Theorem 3.3. *For any $w \in \mathcal{P}$, an optimal strategy for the control problem* (3.2) *is the multi dividend-bands strategy $\pi_{\underline{a}_*,\underline{b}_*}$ and*

$$v_*(x) = V_{\underline{a}_*,\underline{b}_*}(x) := \begin{cases} W^{(q)}(x)\, C^* + F_w(x), & x \in [0, b^*_{1,+}], \\ x - b^*_{i,-} + V_{\underline{a}_*,\underline{b}_*}(b^*_{i,-}), & x \in (b^*_{i,+}, a^*_{i+1}), \\ F_{f_i}(x - a^*_i), & x \in [a^*_i, b^*_{i,+}], i > 1, \end{cases} \tag{3.4}$$

where $f_i : \mathbb{R}_- \to \mathbb{R}$ is given by $f_i(x) = V_{\underline{a}_*,\underline{b}_*}(a^*_i + x)$,

$$C^* = \frac{1 - F'_w(b^*_{1,+})}{W^{(q)\prime}(b^*_{1,+})} \vee \frac{1 - F'_w(b^*_{1,+}-)}{W^{(q)\prime}(b^*_{1,+}-)},$$

and the levels $\underline{a}_*$, $\underline{b}^-_*$ and $\underline{b}^+_*$ are specified in Sect. 5.4 below.

The construction of the strategy $\pi_{\underline{a}_*,\underline{b}^\pm_*}$ and the proof of its optimality are given in Sect. 4 and 5.

4. Martingale approach

4.1. **Dual representation.** The solution of the stochastic control problem (3.2) is based on a characterization of the optimal value function v_*, as the point-wise minimum of the following family of functions:

Definition 4.1. A function $g : \mathbb{R} \to \mathbb{R}$ is a *stochastic super-solution* for the stochastic control problem (3.2) if $g|_{\mathbb{R}_+}$ is upper-semi-right-continuous (USRC)[1],

$$\left\{ \mathrm{e}^{-q\left(t\wedge T_0^-\right)} g\left(X_{t\wedge T_0^-}\right),\ t \in \mathbb{R}_+ \right\} \text{ is a UI } \mathbb{P}_x\text{-super-martingale, for any } x \in \mathbb{R}_+, \tag{4.1}$$

and the following conditions are satisfied:

$$g(x) \geq w(x) \qquad \text{for all } x \in \mathbb{R}_-, \tag{4.2}$$

$$g(x+y) - g(x) \geq y - K \qquad \text{for all } x, y \in \mathbb{R}_+. \tag{4.3}$$

The family of such functions will be denoted by $\mathcal{G}$.

Theorem 4.2. (i) *The value function v_* is the smallest stochastic super-solution for the stochastic control problem* (3.2)*:*

$$v_*(x) = \min_{g\in\mathcal{G}} g(x) \qquad \text{for all } x \in \mathbb{R}_+. \tag{4.4}$$

(ii) *Furthermore, if there exist $b > 0$, $\pi \in \Pi$ and $g \in \mathcal{G}$ such that $g(x) = v_\pi(x)$ for all $x \leq b$ then $v_*(x) = v_\pi(x)$ for all $x \in [0, b]$.*

Remark 4.3. (i) The identity in Eqn. (4.4) is reminiscent of the well-known characterization in Dynkin [18] of the value-function of an optimal stopping problem with non-negative continuous reward function h and rate of discounting q as the smallest q-excessive function that dominates h.

(ii) Def. 4.1 is closely to the notion of (stochastic) sub-solution that was introduced by Stroock & Varadhan [44] in the setting of linear parabolic PDEs and their associated diffusions. In the same setting Bayraktar & Sîrbu [11] recently showed that the point-wise supremum of (stochastic) sub-solutions is a lower semi-continuous viscosity sub-solution of a corresponding Cauchy problem.

(iii) It can *a priori* be shown that v_* is continuous if $K = 0$. See Lemma 4.7 below.

(iv) A sufficient condition for a $g \in C^2(\mathbb{R})$ to satisfy Eqn. (4.1) is that ${}_a\mathcal{L}^w_\infty g \leq 0$ where

$${}_a\mathcal{L}^w_\infty : C^2([a,\infty)) \to D([a,\infty)), \qquad a \in \mathbb{R},$$

is a family of operators that is defined as follows:

$$\begin{aligned} {}_a\mathcal{L}^w_\infty f(x) &= \frac{\sigma^2}{2} f''(x) + (\eta - \overline{\nu}_1(x-a)) f'(x) - (q + \overline{\nu}(x-a)) f(x) \\ &+ \int_{(0,x-a]} \left[f(x) - f(x-y) + f'(x)y \right] \nu(\mathrm{d}y) + \int_{(x-a,\infty)} w(x-y)\nu(\mathrm{d}y), \end{aligned} \tag{4.5}$$

[1] g is USRC at $x > 0$ if $\limsup_{x_n\to x, x_n > x} g(x_n) \leq g(x)$.

where $\overline{\nu}(x) = \nu((x,\infty))$ and $\overline{\nu}_1(x) = \int_{(x,\infty)} y\nu(\mathrm{d}y)$. In case that X has bounded variation the operator ${}_a\mathcal{L}^w_\infty$ takes the following equivalent form:

$$\begin{aligned} {}_a\mathcal{L}^w_\infty f(x) &= pf'(x) + \int_{(0,x-a]} [f(x) - f(x-y)]\,\nu(\mathrm{d}y) + \int_{(x-a,\infty)} w(x-y)\nu(\mathrm{d}y) \\ &- (q + \overline{\nu}(x-a))f(x). \end{aligned} \tag{4.6}$$

The operator ${}_a\mathcal{L}^w_\infty$ is related to the infinitesimal generator Γ of the Feller-semigroup of X as follows: Γ acts on functions f in the set $\{f \in C^2_c(\mathbb{R}) : f|_{(-\infty,a)} = w|_{(-\infty,a)}\}$ as $\Gamma f(x) = {}_a\mathcal{L}^w_\infty g(x)$ for $x > a$, where $g = f|_{[a,\infty)}$ (cf. Sato [41, Thm. 31.5]).

4.2. **Properties of the value function.** The proof of the dual representation is based on an alternative representation of v_* as the point-wise minimum of a class of *majorizing* solutions of the stochastic optimal control problem.

Definition 4.4. A Borel-measurable function $H : \mathbb{R} \to \mathbb{R}$ is called a *majorizing solution* for the stochastic control problem (3.2) if the following conditions are satisfied:

$$\mathrm{e}^{-q(\tau^\pi \wedge t)} H(U^\pi_{\tau^\pi \wedge t}) + \int_0^{\tau^\pi \wedge t} \mathrm{e}^{-qs} \mu^\pi_K(\mathrm{d}s) \text{ is a UI } \mathbb{P}_x\text{-super martingale, for any } x \in \mathbb{R}_+,\ \pi \in \Pi, \tag{4.7}$$

$$H(x) \geq w(x) \qquad \text{for all } x \in \mathbb{R}_-, \tag{4.8}$$

where μ^π_K denotes the random measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ defined in Eqn. (3.3). The family of such functions will be denoted by $\mathcal{H}$.

Proposition 4.5. *The value function v_* admits the following dual representation:*

$$v_*(x) = \min_{H \in \mathcal{H}} H(x) \qquad \textit{for all } x \geq 0. \tag{4.9}$$

Remark 4.6. More generally, the value-function v_* restricted to $[z,\infty)$, $z \in \mathbb{R}_+$, admits the following representation:

$$v_*(x) = \min_{H \in \mathcal{H}_z} H(x) \qquad \text{for all } x \geq z,\ z \in \mathbb{R}_+, \tag{4.10}$$

where $\mathcal{H}_z$ is the set of Borel-measurable functions H satisfying $H(x) \geq v_*(x)$ for all $x \leq z$ and

$$\mathrm{e}^{-q(\tau^\pi_z \wedge t)} H(U^\pi_{\tau^\pi_z \wedge t}) + \int_0^{\tau^\pi_z \wedge t} \mathrm{e}^{-qs} \mu^\pi_K(\mathrm{d}s) \text{ is a UI } \mathbb{P}_x\text{-super martingale, for any } x \geq z,\ \pi \in \Pi,$$

where $\tau^\pi_z = \inf\{t \geq 0 : U^\pi_t < z\}$.

The proof of the representations in Eqns. (4.9) and (4.10) is in part based on the fact that the function v_* is itself a member of the class $\mathcal{H}$, which follows from the following auxiliary results (the proofs of which are deferred to the appendix):

Lemma 4.7. *For any fixed $\pi \in \Pi$ and $x \in \mathbb{R}_+$, the process $V^\pi = \{V_t^\pi, t \in \mathbb{R}_+\}$ defined as follows:*

$$V_t^\pi = \mathrm{e}^{-q(\tau^\pi \wedge t)} v_*(U^\pi_{\tau^\pi \wedge t}) + \int_0^{\tau^\pi \wedge t} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s), \tag{4.11}$$

is a $\mathbb{P}_x$-super-martingale, where v_* is extended to the negative half-axis by $v_*(x) = w(x)$ for $x < 0$.

Lemma 4.8. *(i) For every $x, y \geq 0$, with $x \geq y$, it holds that*

$$(x - y + K) \leq v_*(x) - v_*(y) \leq \left(1 - \frac{W^{(q)}(y)}{W^{(q)}(x)}\right) [v_*(x) - F_w(x)] + F_w(x) - F_w(y). \tag{4.12}$$

In particular, $v_*|_{\mathbb{R}_+}$ is USRC. Furthermore, $v_*|_{\mathbb{R}_+}$ is continuous if $K = 0$.

(ii) v_* is dominated by an affine function: $K + v_*(0) \leq v_*(x) - x \leq \frac{1}{\Phi(q)}$ for all $x \in \mathbb{R}_+$.

Lemma 4.9. *For any $q > 0$, and $w \in \mathcal{P}$, there exists a $C \in \mathbb{R}_{++}$ such that following bound hold true:*

$$\sup_{x \in \mathbb{R}_+} \sup_{\pi \in \Pi} \mathbb{E}_x \left[\mathrm{e}^{-q\tau} w(U_\tau^\pi)\right] \geq -C. \tag{4.13}$$

Furthermore, for any $\pi \in \Pi$ and $x \in \mathbb{R}_+$, the following holds true:

$$\mathbb{E}_x \left[\sup_{t \in \mathbb{R}_+} \left\{\mathrm{e}^{-qt} U_t^\pi \mathbf{1}_{\{t < \tau^\pi\}} + \int_0^t \mathrm{e}^{-qs} \mathrm{d}D_s^\pi + \int_0^t \mathrm{e}^{-qs} X_s \mathrm{d}s\right\}\right] < \infty. \tag{4.14}$$

In particular, the process $V^\pi = \{V_t^\pi, t \in \mathbb{R}_+\}$ defined in Eqn. (4.11) *is UI under $\mathbb{P}_x$.*

Proof of Prop. 4.5: Fix $x \in \mathbb{R}_+$, and let H be any element of $\mathcal{H}$, and $\pi \in \Pi$ any admissible policy. The super-martingale property (4.7), the boundary condition (4.8), and the uniform integrability yield the following:

$$\begin{aligned} H(x) \quad &\geq \quad \lim_{t \to \infty} \mathbb{E}_x \left[\mathrm{e}^{-q(\tau^\pi \wedge t)} H(U^\pi_{\tau^\pi \wedge t}) + \int_0^{\tau^\pi \wedge t} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s)\right] \\ &\geq \quad \mathbb{E}_x \left[\mathrm{e}^{-q\tau^\pi} w(U^\pi_{\tau^\pi}) + \int_0^{\tau^\pi} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s)\right]. \end{aligned}$$

Taking the supremum over $\pi \in \Pi$ and using the definition of v_* yields that $H(x) \geq v_*(x)$. Since $H \in \mathcal{H}$ was arbitrary, it holds thus that $\inf_{H \in \mathcal{H}} H(x) \geq v_*(x)$. This inequality is in fact an equality since v_* is a member of $\mathcal{H}$, on account of Lem. 4.7 and Lem. 4.9. □

The following result implies that the set $\mathcal{H}$ contains the set $\mathcal{G}$:

Lemma 4.10 (Shifting Lemma)**.** *Let $g \in \mathcal{G}$ be dominated by an affine function. If*

$$\left\{\mathrm{e}^{-q(t \wedge T_0^-)} g(X_{t \wedge T_0^-}), t \in \mathbb{R}_+\right\} \text{ is a } \mathbb{P}_x\text{-super-martingale for all } x \in \mathbb{R}_+,$$

then, for any $\pi \in \Pi$ and $x \in \mathbb{R}_+$, $M^\pi = \{M_t^\pi, t \in \mathbb{R}_+\}$ is a $\mathbb{P}_x$-super martingale, where

$$M_t^\pi = \mathrm{e}^{-q(t \wedge \tau^\pi)} g(U^\pi_{t \wedge \tau^\pi}) + \int_0^{\tau^\pi \wedge t} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s). \tag{4.15}$$

Proof of Thm. 4.2: (i) The identity (4.4) follows from Prop. 4.5 in view of the observations that (a) $\mathcal{G}$ is contained in $\mathcal{H}$ and (b) v_* is an element of the set $\mathcal{G}$. Observation (a) follows on account of Lem. 4.10, while observation (b) is a direct consequence of Lem. 4.7 (taking π equal to the strategy $\pi \equiv 0$ of not paying any dividends) and Lem. 4.8(i). The statement (ii) follows by combining part (i) with the definition of v_*. □

Proof of Lem. 4.10: Fix arbitrary $X_0 = x \in \mathbb{R}_+$ and $\pi \in \Pi$ and $s, t \in \mathbb{R}_+$ with $s \leq t$. Since g and h are USRC, M_t^π is $\mathcal{F}_t$-measurable. Furthermore, M_t^π is integrable in view of Eqn (4.14) and the fact that g and h are dominated by an affine function. Consider the sequence $(\pi_n)_{n\in\mathbb{N}}$ of strategies defined by $\pi_n = \{D_t^{\pi_n}, t \in \mathbb{R}_+\}$ with

$$D_u^{\pi_n} = \begin{cases} \sup\{D_v^\pi : v \leq u, v \in \mathbb{T}_n\}, & u < \tau^\pi, \\ D_{\tau^\pi}^\pi, & u \geq \tau^\pi, \end{cases} \qquad \mathbb{T}_n := \left\{t - (t-s)\frac{k}{2^n}, k \in \mathbb{Z}\right\} \cap \mathbb{R}_+.$$

We claim that for every $n \in \mathbb{N}$

$$M^{(n)} = \{M_u^{\pi_n} : u \in \mathbb{T}_n\} \text{ is a } \mathbb{P}_x\text{-martingale.} \tag{4.16}$$

Given this claim the proof is completed as follows. On account of the form of π_n, it follows that $D^{\pi_n} \nearrow D^\pi$ as $n \to \infty$, which, combined with the Monotone Convergence Theorem (MCT) and an integration-by-parts, implies that $\int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}D_s^{\pi_n} \nearrow \int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}D_s^\pi$ and, if $K > 0$, $\int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}N_s^{\pi_n} \nearrow \int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}N_s^\pi$. The fact that the grid $\mathbb{T}_n$ contains both s and t, the MCT and the martingale property (4.16) imply the following:

$$\mathbb{E}[M_t^\pi|\mathcal{F}_{s\wedge\tau^\pi}] = \lim_{n\to\infty} \mathbb{E}[M_t^{\pi_n}|\mathcal{F}_{s\wedge\tau^\pi}] \leq \lim_{n\to\infty} M_{s\wedge\tau^\pi}^{\pi_n} = M_{s\wedge\tau^\pi}^\pi = M_s^\pi.$$

Since s, t were arbitrary it follows that M^π is a $\mathbb{P}_x$-martingale.

Next we turn to the proof of the claim. Denoting $T_i := \tau^\pi \wedge t$ and $M = M^{(n)}$, $D = D^{\pi_n}$ we can write

$$M_t - M_0 = \sum_{i=1}^n Y_i + \sum_{i=1}^n Z_i,$$

where $Z_i = \mathrm{e}^{-qT_i}(g(X_{T_i} - D_{T_i}) - g(X_{T_i} - D_{T_{i-1}}) - \Delta D_{T_i} + K)$ with $\Delta D_{T_i} = D_{T_i} - D_{T_{i-1}}$ and

$$Y_i = \mathrm{e}^{-qT_i}g(X_{T_i} - D_{T_{i-1}}) - \mathrm{e}^{-qT_{i-1}}g(X_{T_{i-1}} - D_{T_{i-1}}).$$

In view of Eqn. (4.3) it follows that Z_i is non-positive. Furthermore, the discrete time version of Doob's stopping theorem and the strong Markov property of X imply the following identity:

$$\mathbb{E}_x[Y_i|\mathcal{F}_{T_{i-1}}] = \mathrm{e}^{-qT_{i-1}}\mathbb{E}_y\left[\mathrm{e}^{-q\tau_i}g(X_{\tau_i}) - g(X_0)\right]\Big|_{y=U_{T_{i-1}}} \leq 0,$$

where $\tau_i = T_i \circ \theta_{T_{i-1}}$ with θ denoting the translation-operator. The tower-property of conditional expectation then yields

$$\mathbb{E}_x[M_t^\pi - M_s^\pi|\mathcal{F}_s] \le \sum_{i=1}^n \mathbf{1}_{\{T_{i-1}>s\}}\mathbb{E}_x\left[\mathbb{E}_x[Y_i|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_s\right] \le 0.$$

Hence, $M = M^{(n)}$ is a super-martingale. □

4.3. **Martingale pasting.** The optimality of the value-function g of a candidate-optimal policy satisfying the bound in Eqn. (4.3) will follow from Thm. 4.2 once the super-martingale property of $M(t) = \exp(t\wedge T_0^-)g(X(t\wedge T_0^-))$ is established. In the following result it is shown that this verification can be carried out *locally* in the sense that M is a super-martingale, provided that the function g is sufficiently regular and that M^τ is a super-martingale for a suitable localization τ:

Lemma 4.11 (Pasting Lemma)**.** *Let $(C_i)_{i\in\mathbb{N}\cup\{0\}}$ be a sequence in $\mathbb{R}_+$ with $C_0 = 0$ and $C_{i-1} < C_i$ for all $i \in \mathbb{N}$, and let $(\tau_i)_i$ be the following sequence of stopping times:*

$$\tau_i = \inf\{t \in \mathbb{R}_+ : X_t \notin [C_{i-1}, C_i)\}, \qquad i \in \mathbb{N}.$$

Let $h : \mathbb{R}_+ \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions bounded by affine functions and assume that $g|_{\mathbb{R}_{++}}$ is continuous if X has bounded variation, and is C^1 if X has unbounded variation and define

$$Y = \left\{\mathrm{e}^{-q(t\wedge T_0^-)}g(X_{t\wedge T_0^-}), t \in \mathbb{R}_+\right\}, \tag{4.17}$$

If the stopped processes $Y^{\tau_i} = \{Y_{t\wedge\tau_i}, t \in \mathbb{R}_+\}$ are $\mathbb{P}_x$-super-martingales, for any $x \in [C_{i-1}, C_i)$ and $i \in \mathbb{N}$, then Y is a $\mathbb{P}_x$-super-martingale for any $x \in \mathbb{R}_+$.

Proof: For the ease of presentation we will restrict ourselves to the case of a partition of the form $[0, a) \cup [a, \infty)$ for $a > 0$. The general case follows by a similar line of reasoning.

Denote $M = Y^{T_0^-}$ and fix $t > 0$ and $x \in \mathbb{R}_+$. Suppose first that X has bounded variation. Then 0 is irregular for $(-\infty, 0)$ for X, so that the following set of stopping times forms a discrete set:

$$T_0 = 0, \qquad T_{2i} = (T_a^+ \wedge T_0^-)\circ\theta_{T_{2i-1}}, \qquad T_{2i-1} = T_a^- \circ \theta_{T_{2i-2}}, \qquad i \in \mathbb{N}, \tag{4.18}$$

where θ denotes the translation operator. The strong Markov property of X implies that the following holds on the event $\{s \le T_{i-1}\}$:

$$\begin{aligned}\mathbb{E}[M_{t\wedge T_i} - M_{t\wedge T_{i-1}}|\mathcal{F}_s] &= \mathbb{E}\left[\mathbb{E}[M_{t\wedge T_i} - M_{t\wedge T_{i-1}}|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_s\right]\\ &= \mathbb{E}\left[1_{\{t>T_{i-1}\}}\mathrm{e}^{-qT_{i-1}}\mathbb{E}_{X_{T_{i-1}}}\left[\mathrm{e}^{-qR_v}g(X_{R_v}) - g(x)\right]\Big|_{v=T_{i-1}}\middle|\mathcal{F}_s\right],\end{aligned} \tag{4.19}$$

where $R_v = (\tau' \wedge t)\circ\theta_v$ where τ' is set equal to $T_{0,a}$ if $X_0 = x \in [0, a)$ and to T_a^- if $X_0 = x \ge a$. The expectation in (4.19) is zero in view of Doob's optional stopping theorem and the super-martingale

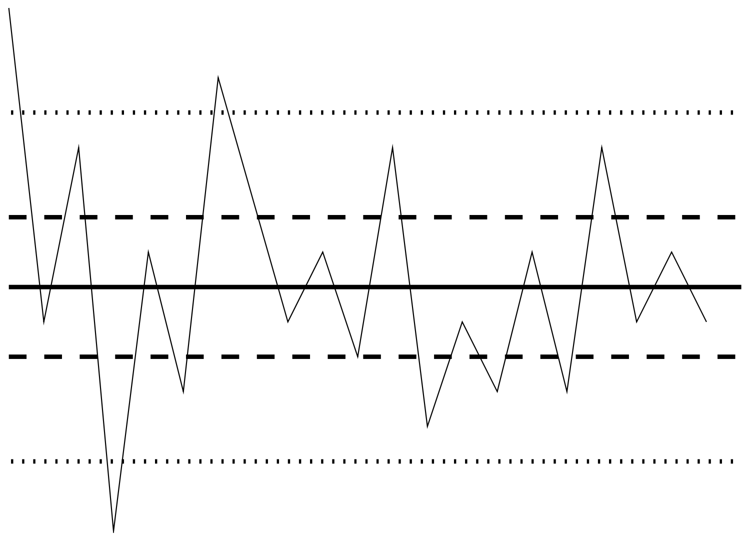

FIGURE 2. The martingale increments commence when X enters the inner band (dashed) and stop when X leaves the outer band (dotted).

property in Eqn. (4.17). The stated super-martingale property then follows on account of the fact that the terms in the following sum have non-positive expectation under $\mathbb{E}[\cdot|\mathcal{F}_s]$:

$$M_t - M_s = \sum_j 1_{\{T_{j-1} < s \leq T_j\}} \left\{ (M_{T_j \wedge t} - M_{T_j \wedge s}) + \sum_{i > j} (M_{t \wedge T_i} - M_{t \wedge T_{i-1}}) \right\}, \qquad s < t.$$

Suppose now that X has unbounded variation. Denote by $(T_i)_{i \in \mathbb{N} \cup \{0\}}$ the sequence of subsequent passage times into the sets $[a - \epsilon, a + \epsilon]$ and $\mathbb{R} \backslash [a - 2\epsilon, a + 2\epsilon]$:

$$T_0 = 0, \qquad T_{2i-1} = H_{[a-\epsilon, a+\epsilon]} \circ \theta_{T_{2i-2}} \qquad T_{2i} = T_{a-2\epsilon, a+2\epsilon} \circ \theta_{T_{2i-1}} \qquad i \in \mathbb{N},$$

where, for any Borel set A, $H_A = \inf\{t \in \mathbb{R}_+ : X_t \in A\}$ (see Figure 4.3). Decompose M as $M - M_0 = M^{(1)} + M^{(2)}$, where

$$M_t^{(1)} = \sum_{i \geq 1} [M_{t \wedge T_{2i}} - M_{t \wedge T_{2i-1}}], \qquad M_t^{(2)} = \sum_{i \geq 1} [M_{t \wedge T_{2i-1}} - M_{t \wedge T_{2i-2}}].$$

The sum $M^{(1)}$ of increments of M during the periods that X spends in the band $[a - 2\epsilon, a + 2\epsilon]$ vanishes in expectation as $\epsilon \searrow 0$, as shown in the following result the proof of which is given in the Appendix:

Lemma 4.12. *As* $\epsilon \searrow 0$, $\mathbb{E}_x[|M_t^{(1)}|] \to 0$.

By the line of the reasoning given in the first part of the proof it follows that $M^{(2)}$ is a super-martingale for every $\epsilon > 0$, so that also M is a super-martingale in view of Lem. 4.12. $\square$

4.4. **Value-function for large levels of the reserves.** From the form of the generator it can be deduced that the value-function is affine for large levels of the reserves:

Proposition 4.13. *Suppose that either (i) $K = 0$ or (ii) $K > 0$ and $\nu(\mathbb{R}_+) < \infty$. Then, for sufficiently large levels of the reserves, it is optimal to immediately pay out a lump-sum dividend, and for some $y \in \mathbb{R}_+$, the function v_* restricted to $[y, \infty)$ takes the following form:*

$$v_*(x) = x - y + v_*(y) \text{ for any } x - y \in \mathbb{R}_+. \tag{4.20}$$

The proof rests on the fact that the generator ${}_y\mathcal{L}_\infty^{v_*}$ applied to the function $\ell_y : [y, \infty) \to \mathbb{R}$ defined by $\ell_y(x) = x - y + v^*(y)$ will be negative for y sufficiently large:

Lemma 4.14. *Suppose that either (i) $K = 0$ or (ii) $K > 0$ and $\nu(\mathbb{R}_+) < \infty$. For $y \in \mathbb{R}_+$ sufficiently large, the following holds true:*

$$ {}_y\mathcal{L}_\infty^{v_*}(\ell_y)(x) \leq 0 \qquad \forall x > y. \tag{4.21}$$

An application of Itô's lemma (which is justified since $\ell_y \in C^2([y, \infty)))$ yields the following fact:

$$\mathrm{e}^{-q(t\wedge T_y^-)}\ell_y(X_{t\wedge T_y^-}) - \int_0^{t\wedge T_y^-} \mathrm{e}^{-qs}\,{}_y\mathcal{L}_\infty^{v_*}(\ell_y)(X_s)\mathrm{d}s \text{ is a martingale,} \tag{4.22}$$

so that $\{\mathrm{e}^{-q(t\wedge T_y^-)}\ell_y(X_{t\wedge T_y^-}), t \in \mathbb{R}_+\}$ is a super-martingale. Hence, the assertion in Prop. 4.13 follows from Rem. 4.6.

Proof of Lem. 4.14: We show that the criterion (4.21) is satisfied. Observe that, for any $x \in \mathbb{R}_+$,

$$\begin{aligned} {}_y\mathcal{L}_\infty^{v_*}\ell_y(x) &= \eta\ell_y'(x) - q\ell_y(x) + \int_{(0,\infty)} [\ell_y(x-z) - \ell_y(x) + z\ell_y'(x)]\nu(\mathrm{d}z) \\ &= \eta - q(x - y + v_*(y)) + \int_{(x-y,\infty)} [v_*(x-z) - v_*(y) + z + y - x]\nu(\mathrm{d}z). \end{aligned} \tag{4.23}$$

Since $v_*(x-z)-v_*(y) \leq x-z-y+K$ for $z \in (x-y, x)$ on account of Lem. 4.8(ii) and w is non-positive, it follows that the integral term in (4.23) is bounded above by $K\nu(x-y, x)+\int_{[x,\infty)}(z+y-x-v^*(y))\nu(\mathrm{d}z)$. In view of Lem. 4.8(ii) and the fact that the measure $\int_{\{|x|>1\}} x\nu(\mathrm{d}x)$ has finite mass, the integral tends to zero as x and y tend to infinity such that $x - y$ is kept constant. On account of Lem. 4.8(ii) $v_*(y) \to \infty$ as $y \to \infty$, so that it follows that ${}_y\mathcal{L}_\infty^{v_*}\ell_y(x)$ is strictly negative for all y sufficiently large and all $x > y$. □

5. Construction of the optimal value function

5.1. **Single dividend-band strategies.** We will first consider the case of single dividend band strategies. The value $v_b(x) := v_{\pi_b}(x)$ associated to the single dividend band strategy π_b at a non-zero level b when X_0^b is equal to x, is given by

$$v_b(x) = \mathbb{E}_x\left[\int_0^{\sigma_b} \mathrm{e}^{-qt}\mu_K^b(\mathrm{d}t) + \mathrm{e}^{-q\sigma_b}w(U_{\sigma_b}^b)\right], \tag{5.1}$$

where $\mu_K^b := \mu_K^{\pi_b}$, $U^b := U^{\pi_b}$ and $\sigma_b := \tau^{\pi_b} = \inf\{t \in \mathbb{R}_+ : U_t^{b+} < 0\}$. In the following result v_b is explicitly expressed in terms of scale functions.

Proposition 5.1. *For $b_+ > b_- \geq 0$ and $x \in [0, b_+]$ it holds that*

$$v_b(x) = \begin{cases} w(x), & x < 0, \\ \\ W^{(q)}(x)\ G(b_-, b_+) + F(x), & x \in [0, b_+], \end{cases} \tag{5.2}$$

where $F = F_w$ and

$$G(b_-, b_+) := \begin{cases} \dfrac{b_+ - b_- - K - (F(b_+) - F(b_-))}{W^{(q)}(b_+) - W^{(q)}(b_-)}, & K > 0,\ b_+ > b_-, \\ \\ \dfrac{1 - F'(b_+)}{W^{(q)\prime}(b_+)} =: G^{\#}(b_+), & K = 0,\ b_+ = b_-. \end{cases} \tag{5.3}$$

Proof. Let $K > 0$. Taking note of the fact that no dividend payment takes place before X reaches the level b_+ it follows that $\{X_t, t \leq T_{0,b_+}\}$ and $\{U_t^{b+}, t \leq \sigma_{b_+}\}$ have the same law. In view of the strong Markov property of X and the absence of positive jumps it follows then that for all $x \in [0, b_+]$ and with $v = v_b$:

$$\begin{aligned} v(x) &= \mathbb{E}_x[\mathrm{e}^{-qT_{b_+}^+}(v(b_-) + \Delta b - K)\mathbf{1}_{\{T_{b_+}^+ < T_0^-\}}] + \mathbb{E}_x[\mathrm{e}^{-qT_0^-} w(U_{T_0^-})\mathbf{1}_{\{T_{b_+}^+ > T_0^-\}}] \\ &= \frac{W^{(q)}(x)}{W^{(q)}(b_+)}[v(b_-) + \Delta b - K] + \left[F(x) - F(b_+)\frac{W^{(q)}(x)}{W^{(q)}(b_+)}\right], \end{aligned} \tag{5.4}$$

where $F = F_w$ and we used the identity (2.11), and the definition of a single dividend band strategy. Evaluating Eqn. (5.4) at $x = b_-$, solving the resulting linear equation for $v(b_-)$ and inserting the result in Eqn. (5.4) yields Eqn. (5.2). The case $K = 0$ follows by a similar line of reasoning. □

We next turn to the determination of the candidate optimal levels. The form of G suggests to define the level $b^* = (b_-^*, b_+^*)$ as the maximizer of $G(x, y)$ over all $x, y \geq 0$ if $K > 0$, and similarly, to define b_+^* as the maximizer $G(x)$ over all $x \geq 0$, if $K = 0$.

Remark 5.2. On account of the facts that the map $x \mapsto G^{\#}(x)$ is right-continuous and monotone decreasing for all x sufficiently large (Prop. D.3), there exists an $x^* \in \mathbb{R}_+$ such that $\sup_{x \geq 0} G^{\#}(x) = G^{\#}(x^*) \vee G^{\#}(x^*-)$. In the case that K is strictly positive, G attains its maxmimum at some $(x^*, y^*) \in (0, \infty)^2$, since $G(x, y)$ is continuous at any (x, y) with $y > x \geq 0$, is monotone decreasing for y sufficiently large and fixed x (Prop. D.3, Appendix D) and tends to minus infinity if $x \searrow b_-$ and tends to the constant κ_w in Eqn. (2.26) if $|x| + |y| \nearrow \infty$ such that $x < y$.

Moreover, if K is strictly positive, observe that the partial right-derivatives of $G(x, y)$ are given by

$$\frac{\partial G}{\partial x}(x, y) = \frac{W^{(q)\prime}(x)}{W^{(q)}[x, y]}[G(x, y) - G^{\#}(x)], \quad \frac{\partial G}{\partial y}(x, y) = -\frac{W^{(q)\prime}(y)}{W^{(q)}[x, y]}[G(x, y) - G^{\#}(y)], \tag{5.5}$$

where $W^{(q)}[x,y] := W^{(q)}(y) - W^{(q)}(x)$ and $G^{\#}(x)$ is given in Eqn. (5.3) Therefore, an interior maximum (x^*, y^*) will satisfy $G(x^*, y^*) = G^{\#}(x^*) = G^{\#}(y^*)$, and a candidate optimum may be found by fixing $d = y - x$, and optimizing the left end-point $x(d)$ for fixed d (graphically, this would amount to determining the highest value of the function $G^{\#}$ where the "width" $y(d) - x(d)$ of the function $G^{\#}$ is d).

In the case of strictly positive K we fix therefore $d > 0$ and set

$$b^*(d) = \sup\{b \geq 0 : G(b, b+d) \geq G(x, x+d) \quad \forall x \geq 0\}. \tag{5.6}$$

and define d^* as follows:

$$d^* = \inf\{d > 0 : G(b^*(d), b^*(d) + d) \geq G(b^*(y), b^*(y) + y) \quad \forall y \geq 0\}. \tag{5.7}$$

The candidate optimal levels are then defined as follows:

$$b^* = (b_-^*, b_+^*) \quad \text{with} \quad b_-^* = b^*(d^*), \quad b_+^* = b^*(d^*) + d^*. \tag{5.8}$$

In the absence of transaction cost ($K = 0$), we set

$$b_+^* = b_-^* = \inf\{b \geq 0 : G(b) \vee G(b-) \geq G(x) \ \ \forall x \geq 0\}, \tag{5.9}$$

where we denote $G(0-) = G(0)$.

A necessary and sufficient condition for the optimality of the policy π_{b^*} can be explicitly expressed in terms of the function $G^* : (b_-^*, \infty) \to \mathbb{R}$ that is defined as follows:

$$G^*(y) := G(b_-^*, y) = \begin{cases} \dfrac{y - b_-^* - K - (F(y) - F(b_-^*))}{W^{(q)}(y) - W^{(q)}(b_-^*)}, & K > 0, \\[2ex] \dfrac{1 - F'(y)}{W^{(q)\prime}(y)}, & K = 0. \end{cases} \tag{5.10}$$

Remark 5.3. In view of its form the function G^* is left- and right-differentiable at any $y > 0$ if $K > 0$, and is equal to a difference of two monotone real-valued functions if $K = 0$ (cf. Lem. A.2(vi)).

Recall that a function $f : (a, \infty) \to (0, \infty)$, $a \in \mathbb{R}$, is *completely monotone* if $(-1)^{k-1} f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}$ and $x > a$, where $f^{(k)}$ denotes the kth derivative with respect to x.

Theorem 5.4. *(i) For any $x \leq b_+^*$, it holds that $v_*(x) = v_{b^*}(x)$. In particular, if $X_0 \in [0, b_+^*]$, it is optimal to adopt the strategy π_{b^*}.*

(ii) The strategy π_{b^*} is optimal for the stochastic optimal control problem (3.2) *if and only if $\Xi : (\Phi(q), \infty) \to \mathbb{R}$ is completely monotone, where*

$$\Xi(\theta) = -\frac{e^{\theta b_+^*}}{\theta} \int_{(b_+^*, \infty)} e^{-\theta z} Z^{(q,\theta)\prime}(z) G^*(dz). \tag{5.11}$$

(iii) In particular, if G^* is non-increasing on (b_+^*, ∞), then the strategy π_{b^*} is optimal.

Remark 5.5. In the absence of transaction cost ($K = 0$), the function Ξ in (5.11) can be equivalently expressed as

$$\Xi(\theta) = G^*(b_+^*)L_0(\theta) + \frac{(\psi(\theta) - q)}{\theta^2}\, \mathbb{E}[F'(b_+^* + \mathbf{e}_\theta) - F'(b_+^*)],$$

where $\mathbf{e}_\theta$ denotes an independent exponential random variable with mean θ^{-1} and $L_0 : (0, \infty) \to \mathbb{R}$ is given by

$$\theta \mapsto L_0(\theta) := \frac{\psi(\theta) - q}{\theta^2}\, \mathbb{E}[W^{(q)\prime}(b_+^* + \mathbf{e}_\theta) - W^{(q)\prime}(b_+^*)].$$

In particular, if the penalty is zero and there are no transaction cost ($w = K = 0$), the necessary and sufficient optimality condition simplifies to the complete monotonicity of $L_0(\theta)$ on the interval $(\Phi(q), \infty)$. This observation appears new even in this particular case.

Remark 5.6. (i) The function $g_* : \mathbb{R}_+ \to \mathbb{R}$ defined by $g_*(x) = W^{(q)}(x)G^*(b_+^*) + F_w(x)$ is a super-solution in the sense of Def. 4.4, and hence dominates the value-function v_*. In fact, since $g_*(x)$ is equal to the value $v_{b^*}(x)$ of the strategy π_{b^*} for any level x of initial reserve smaller or equal to b_+^*, Thm. 4.2(ii) implies that $g_*(x)$ is equal to the optimal value $v_*(x)$ for all $x \in [0, b_+^*]$. That g_* is a super-solution follows by combining the Shifting Lemma with the facts that (a) $\mathrm{e}^{-q(t\wedge T_0^-)}g_*(X_{t\wedge T_0^-})$ is a martingale and that (b) g_* satisfies the following inequality:

$$g_*(x) - g_*(y) \geq x - y - K \qquad \text{for any } 0 \leq y < x.$$

The fact (a) in turn follows from the martingale properties of F_w and $W^{(q)}$, while (b) follows on account of the definitions of b^* and G^*: if $K = 0$, the following holds true:

$$g_*'(x) = W^{(q)\prime}(x)G^*(b^*) - F_w'(x) \geq W^{(q)\prime}(x)G^*(x) - F_w'(x) = 1, \qquad x > 0.$$

Similarly, if $K > 0$ and $x > y$, $g_*(x) - g_*(y) = (W^{(q)}(x) - W^{(q)}(y))G(b_-^*, b_+^*) - F_w(x) + F_w(y)$ is bounded below by

$$(W^{(q)}(x) - W^{(q)}(y))G(y, x) - F_w(x) + F_w(y) = x - y - K.$$

The two displays imply that $g_*(x) - g_*(y) \geq x - y - K$ for any $x, y, K \geq 0$ with $x \geq y$. This completes the proof of Thm. 5.4 (i); the proofs of parts (ii) and (iii) are given in Appendix D.

(ii) The strategy π_{b^*} is optimal if and only if the following condition for v_{b^*} is satisfied:

$$ {}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*}(x) \;\leq\; 0, \qquad \text{for all } x > b_+^* \text{ and with } \overline{w} = v_{b^*}, \tag{5.12}$$

where the operator ${}_{b_+^*}\mathcal{L}_\infty^{\overline{w}}$ is defined in (4.5). The necessity of the condition (5.12) follows in view of part (i) and Prop. 4.13. To see that the condition (5.12) is also sufficient, suppose that the condition in Eqn. (5.12) is not satisfied. Since $x \mapsto {}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*}(x)$ is right-continuous for $x \geq b_+^*$, it follows that there exists an open interval (α, β) contained in (b_+^*, ∞) such that ${}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*}(x) > 0$. Define a strategy $\tilde{\pi}$ as follows: whenever U_t does not take a value in the interval (α, β) operate according to π_{b^*}, and while the reserve process U_t takes a value in the interval (α, β), do not pay any dividends. Then

$S_t := \mathrm{e}^{-q(t\wedge T_{\alpha,\beta})}[v_{\tilde{\pi}}(X_{t\wedge T_{\alpha,\beta}}) - v_{b^*}(X_{t\wedge T_{\alpha,\beta}}))$ is a super-martingale and the following holds true (cf. Eqn. (4.22)):

$$\mathbb{E}_x[S_t - S_0] = -\mathbb{E}_x\left[\int_0^{t\wedge T_{\alpha,\beta}} \mathrm{e}^{-qs}{}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*}(X_s)\mathrm{d}s\right] < 0 \qquad \text{for any } x \in (\alpha,\beta).$$

This identity implies that $v_{\tilde{\pi}}(x)$ is strictly larger than $v_{b^*}(x)$ for any $x \in (\alpha,\beta)$.

Explicit conditions can be identified in terms of the penalty w and the Lévy measure ν that guarantee that $G^\#$ is non-increasing on (b_+^*,∞), which are hence sufficient conditions for the optimality of the policy π_{b^*}.

In the absence of transaction cost ($K = 0$) we will call a penalty $w \in \mathcal{R}$ *severe* if (i) $w(0) \le \gamma_w := v_0(0)$, and (ii) $w(x+y) - w(y) \le x$ for all $x, y \in \mathbb{R}_-$. Condition (i) states that the penalty payment for ruin occurring without shortfall is not smaller than the expected value minus transaction cost of liquidation (i.e. the sum of the expected premium income until the moment of ruin and the expected penalty payment), while condition (ii) implies that the additional penalty payment for an additional shortfall of size x is at least x.

The result is phrased as follows:

Theorem 5.7. *Suppose that $K = 0$ and ν admits a convex density ν'. If in addition (a) the penalty w is* severe*, or (b) ν' is* completely monotone*, then G^* is non-increasing on (b_+^*,∞).*

The proof is given in Appendix D. As application we consider next the case of a single-dividend band strategy with $b_+^* = 0$.

5.2. **Liquidation strategy.** In the absence of transaction cost ($K = 0$), the liquidation strategy π_ℓ is to "pay out all the reserves to the beneficiaries and subsequently pay all the premiums as dividends, until the moment of ruin." Note that $\pi_\ell = \pi_0$, that is, a single dividend-band strategy at level 0. In the case that X is given by the Cramér-Lundberg model, the first jump (claim) arrives after an exponential time T with finite mean λ^{-1}, and the value v_ℓ of the liquidation strategy is equal to

$$\begin{aligned}
v_\ell(x) &= \mathbb{E}_x\left[x + p\int_0^T \mathrm{e}^{-qt}\mathrm{d}t + \mathrm{e}^{-qT}w(\Delta X_T)\right] \\
&= \mathbb{E}_x\left[x + \frac{p}{q}(1-\mathrm{e}^{-qT}) + \mathrm{e}^{-qT}(w(\Delta X_T) - w(0)) + w(0)\mathrm{e}^{-qT}\right] \\
&= \left[x + \frac{p}{\lambda+q} + \frac{1}{\lambda+q}w_\nu(0) + \frac{\lambda}{\lambda+q}w(0)\right] = \left[x + \frac{p + w_\nu(0) + \lambda w(0)}{\lambda+q}\right],
\end{aligned}$$

where $\Delta X_T = X(T) - X(T-)$, and $w_\nu : (0,\infty) \to \mathbb{R}$ is defined by (2.23). If $X_0 = 0$ and X has infinite activity, ruin occurs immediately if strategy π_ℓ is followed, that is, in this case $\tau^{\pi_\ell} = 0$, $\mathbb{P}_0$-a.s. and $v_\ell(x) = x + w(0)$.

Hence, the value of the liquidation strategy is equal to $v_\ell(x) = (x+\gamma_w)\mathbf{1}_{\{x\geq 0\}} + w(x)\mathbf{1}_{\{x<0\}}$ where

$$\gamma_w = v_0(0) = \begin{cases} \frac{1}{q+\overline{\nu}}\left[p + w_\nu(0) + \overline{\nu}\, w(0)\right], & \text{if } \overline{\nu} := \nu(0,\infty) < \infty, \\ w(0), & \text{if } \overline{\nu} = \infty. \end{cases} \tag{5.13}$$

As direct consequence of Theorem 5.4(ii) we get the following sufficient condition for the optimality of the liquidation strategy:

Corollary 5.8. *Let $K = 0$. If G^* is monotone decreasing on $\mathbb{R}_+$, then the strategy π_ℓ is optimal.*

5.3. **Auxiliary optimal stopping problem with dividends.** In order to treat the general case of a general multi-dividend band strategy, we consider in this section first an auxiliary stochastic control problem in which the controls are pairs (τ, π) of dividend strategies $\pi \in \Pi$ and $\mathbf{F}$-stopping times τ. The problem is an extension of (3.2) in which the management of the company in addition to deciding the timing and size of dividend payments also has the option to "wind up the company" at any stopping time τ before the ruin time τ^π upon which a payment $f(U^\pi_{\tau^\pi})$ is received, for some pre-specified reward function $f : \mathbb{R}_+ \to \mathbb{R}$. The value-function of this stochastic control problem is given by the following expression:

$$V^*(x) = \sup_{\tau\in\mathcal{T},\pi\in\Pi} V_{\tau,\pi}(x), \qquad V_{\tau,\pi}(x) = \mathbb{E}_x\left[\int_0^{\tau\wedge\tau_\pi} \mathrm{e}^{-qt}\mu^\pi_K(\mathrm{d}t) + \mathrm{e}^{-q(\tau\wedge\tau_\pi)} f_w(U^\pi_{\tau\wedge\tau_\pi})\right], \tag{5.14}$$

where $f_w : \mathbb{R} \to \mathbb{R}$ is defined by $f_w(x) = f(x)$ for $x > 0$ and $f_w(x) = w(x)$ for $x \leq 0$, and $\mathcal{T}$ denotes the set of $\mathbf{F}$-stopping times. The problem (5.14) can be embedded into the optimal control problem (3.2) for an appropriate choice of he reward function f— see Section 5.4.

In this context a candidate optimal policy is the strategy $(\sigma_a^{\pi_b}, \pi^b)$ to pay out dividends according to π_b and wind up the company ("stop") at the first moment $\sigma_a = \sigma_a^{\pi_b}$ that U^{π_b} falls below a in case the first-passage time σ_a is smaller than the ruin time τ^{π_b}. If the transaction cost K are large, an alternative is given by the strategy $(\sigma_{a,b}^{\pi_\emptyset}, \pi_\emptyset)$ not to pay out any dividends and to stop at the first moment $\sigma_{a,b_+} = \sigma_{a,b_+}^{\pi_\emptyset}$ that the reserves process $X = U^{\pi_\emptyset}$ leaves a finite interval $[a, b_+]$. The values $V_{a,b}(x)$ and $V^\emptyset_{a,b_+}(x)$ associated to the strategies (σ_a, π_b) and $(\sigma_{a,b_+}^{\pi_\emptyset}, \pi_\emptyset)$ when $U^b_0 = x$, are given by the following expressions:

$$V_{a,b_-,b_+}(x) = \mathbb{E}_x\left[\int_0^{\sigma_a} \mathrm{e}^{-qt}\mu^b_K(\mathrm{d}t) + \mathrm{e}^{-q\sigma_a} f(U^b_{\sigma_a})\right] \qquad V^\emptyset_{a,b_+}(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\sigma_a} f(U^\emptyset_{\sigma_{a,b_+}})\right], \tag{5.15}$$

where $\mu^b_K = \mu^{\pi_b}_K$. In the following result, which can be derived by a line of reasoning similar to the one used in the proof of Prop. 5.1, the functions $V_{a,b}$ and $V^\emptyset_{a,b}$ can be explicitly expressed in terms of scale functions and of a family of functions $(y,z) \mapsto G^{(a,K)}(y,z)$, $a, K \geq 0$, that is defined as follows (where

we will suppress the dependence on K and write $G^{(a)}$):

$$G^{(a)}(b_-,b_+)=\begin{cases}\dfrac{b_+-b_--K-(F^{(a)}(b_+-a)-F^{(a)}(b_--a))}{W^{(q)}(b_+-a)-W^{(q)}(b_--a)}, & K>0, b_->a \text{ or } b_-=a=0,\\ \dfrac{f(b_+)-F^{(a)}(b_+-a)}{W^{(q)}(b_+-a)}, & K>0, b_-=a>0,\\ \dfrac{1-F^{(a)\prime}(b_+-a)}{W^{(q)\prime}(b_+-a)}, & K=0,\end{cases}$$

where $F^{(a)}=F_{{}_af_w}$ is the Gerber-Shiu function for pay-off ${}_af_w=f_w(a+\cdot)$.

Proposition 5.9. *For any b_-,b_+,a such that $b_+>b_->a\geq 0$ the following holds true:*

$$V_{a,b_-,b_+}(x) = \begin{cases} f(x), & x\in[0,a],\\ W^{(q)}(x-a)G^{(a)}(b_-,b_+)+F^{(a)}(x-a), & x\in[a,b_+],\\ x-b_++V_{a,b}(b_+), & x\in(b_+,\infty);\end{cases} \tag{5.16}$$

$$V^{\emptyset}_{a,b_+}(x) = \begin{cases} f(x), & x\notin[a,b_+],\\ W^{(q)}(x-a)G^{(a)}(a,b_+)+F^{(a)}(x-a), & x\in[a,b_+].\end{cases} \tag{5.17}$$

Remark 5.10. In the sequel we will denote by $V_{a,b}$ (with $b=(b_-,b_+)$) the function that is equal to V_{a,b_-,b_+} if $b_->a$ and equal to $V^{\emptyset}_{a,b_+}$ if $b_-=a$.

We next turn to the determination of the candidate optimal levels $\beta^*(a)$. Fixing a for the moment, we define $\beta^*(a)$ to be the pair $(\beta^*_-(a),\beta^*_+(a))$ that maximizes $G^{(a)}$, similarly as was done in the case of the single dividend-band strategy. Hence, if $K>0$ we set $\beta^*_+(a)=\beta^*(a,\delta^*(a))+\delta^*(a)$ where

$$\begin{cases}\beta^*(a,d)=\sup\{b\geq a: G^{(a)}(b,b+d)\geq G^{(a)}(x,x+d)\ \ \forall x\geq 0\},\\ \delta^*=\inf\{d>0: G^{(a)}(\beta^*(a,d),\beta^*(a,d)+d)\geq G^{(a)}(\beta^*(a,y),\beta^*(a,y)+y)\ \ \forall y\geq 0\}.\end{cases} \tag{5.18}$$

If $K=0$, we define the levels $\beta^*_+(a)=\beta^*_-(a)$ as follows:

$$\beta^*_+(a)=\beta^*_-(a)=\inf\{b\geq a: G^{(a)}(b)\vee G^{(a)}(b-)\geq G^{(a)}(x)\ \ \forall x\geq 0\}.$$

Finally, the level α^* is specified as follows:

$$\alpha^*=\inf\{a\geq 0: G^{(a)}(\beta^*-)\vee G^{(a)}(\beta^*)=0\}, \tag{5.19}$$

where $\inf\emptyset=+\infty$, and we denote $\beta^*:=\beta^*(\alpha^*)=(\beta^*_-(\alpha^*),\beta^*_+(\alpha^*))$. We will write $\alpha^*=\alpha^*_{f_w}$ and $\beta^*=\beta^*_{f_w}$ if we wish to express the dependence of α^* and β^* on f_w.

Remark 5.11. The choice of α^* is informed by the *principles of continuous* and *smooth fit* of the theory of optimal stopping (see [37, Ch. IV.9]) that state that it can be expected that V^* be *continuous* and *continuously differentiable* at the level α^*, respectively. The latter heuristic is in force if α^* is regular

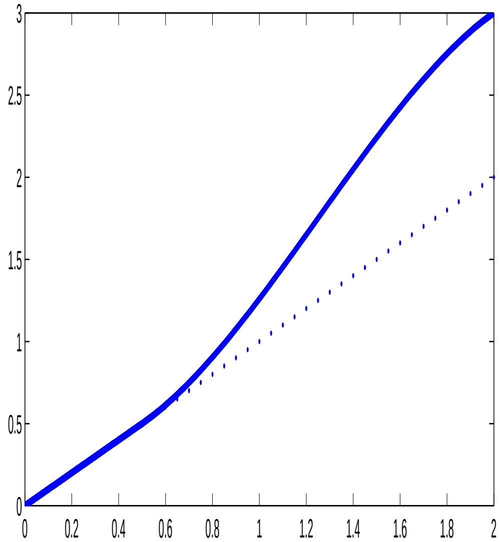

FIGURE 3. Plotted is a typical value-function $V_{\alpha^*,\beta^*}(x)$ for $x \in [0,2]$, $\alpha^* = 0.5$ and $\beta^* = 2$ in the case of no transaction cost ($K = 0$), showing smooth fit of the value-function $V_{\alpha^*,\beta^*}(x)$ with the pay-off function $f(x) = x$ at $x = 0.5$.

for $(-\infty, \alpha^*)$ for U^{π_*}, where π_* denotes the optimal strategy, while the former applies if α^* is irregular for $(-\infty, \alpha^*)$. In view of the fact that α^* is regular for $(-\infty, \alpha^*)$ iff X has unbounded variation, the heuristic implies that, if $\sigma > 0$ or $\int_0^1 x\nu(\mathrm{d}x) = \infty$, then α^* satisfies $V'_{a^*,\beta^*}(a^*+) = f'(a^*-)$. This equation is equivalent to the expression (5.19) on account of the form of $V_{a,b}$ and the facts that $F'_{{}_af}(0+) = f'(a-)$ for any $a > 0$ and $W^{(q)\prime}(0+) \in (0, \infty]$. In the complementary case that X has bounded variation, α^* is irregular for $(-\infty, \alpha^*)$, and the heuristic implies that $V_{\alpha^*,\beta^*}(\alpha^*) = f(\alpha^*)$. This equation can also be equivalently expressed as Eqn. (5.19), in view of the form of $V_{a,b}$ and the fact that $W^{(q)}(0) > 0$ iff X has bounded variation.

We will present a solution to the control problem (5.14) in the following setting:

Assumption 1. *(i) The reward function $f : \mathbb{R}_+ \to \mathbb{R}$ is given by $f(x) = x + c$ for some constant $c \in \mathbb{R}$.*

(ii) There exists a $u > 0$ such that ${}_0\mathcal{L}^w_\infty f(u) > 0$.

(iii) For all $b_+ \geq b_- > 0$ it holds that $G^{(0)}(b_-, b_+) < 0$.

Remark 5.12. (i) If the reward function f is chosen as in As. 1(i) with $c = \gamma_w$ the value of the stochastic control problems in Eqns. (5.14) and (3.2) coincide. This assertion follows as a consequence of the fact that the value of any given admissible strategy (τ, π), $\tau \in \mathcal{T}, \pi \in \Pi$, in the stochastic control problem (5.14) is equal to the value of the strategy $\tilde{\pi} = \{D_t^{\tilde{\pi}}, t \in \mathbb{R}_+\} \in \Pi$ for the stochastic control

problem (3.2) that is defined in terms of the liquidation strategy π_ℓ as follows:

$$D_t^{\tilde\pi} = \begin{cases} D_t^\pi, & \text{if } t < \tau \wedge \tau^\pi, \\ D_\tau^\pi + D^{\pi_\ell} \circ \theta_\tau, & \text{if } \tau \le \tau^\pi. \end{cases}$$

(ii) Under As. 1(i,ii), it follows from Remark 5.6(ii) that if ${}_0\mathcal{L}_\infty^w f_w(y)$ is non-positive for all $y > 0$ then it is not optimal to stop immediately ($\tau^* \not\equiv 0$).

(iii) In the absence of transaction cost ($K = 0$), the condition in As. 1(iii) implies that the optimal single dividend-band strategy defined in Eqn. (5.9) is the one at level $b_+^* = b_-^* = 0$ (cf. Prop. D.3).

Under As. 1, it holds that the levels $\beta_+^*(\alpha^*)$ and α^* are positive and finite:

Lemma 5.13. *Under As. 1 it holds that* $0 < \alpha^* \le \beta_+^*(\alpha^*) < \infty$ *and* $G^{(\alpha^*)}(\beta^*-) \vee G^{(\alpha^*)}(\beta^*) = 0$. *If either (i)* $K > 0$ *or (ii)* $K = 0$ *and* X *has unbounded variation, then* $\alpha^* < \beta_+^*(\alpha^*)$.

The proof of Lem. 5.13 is given at the end of the section.

Define the strategy (τ_*, π_*) to be equal to $(\sigma_{\alpha^*}, \pi_{\beta^*})$ if $\alpha^* < \beta_-^*$ and to be equal to $(\sigma_{\alpha^*,\beta_+^*}, \pi^\emptyset)$ if $\alpha^* = \beta_-^*$, and note that V_{τ_*,π_*} is given explicitly given as follows:

$$V_{\tau_*,\pi_*}(x) = F^{(\alpha^*)}(x - \alpha^*) \qquad \text{for } x \in [\alpha^*, \beta_+^*].$$

A necessary and sufficient condition for optimality of this strategy is as follows:

Theorem 5.14. *Let As. 1 hold. (i) For any* $x \in [0, \beta_+^*]$, *it holds that* $V^*(x) = V_{\alpha^*,\beta^*}(x)$. *In particular, if* $X_0 \in [0, \beta_+^*]$, *it is optimal to adopt the policy* (τ_*, π_*).

(ii) The strategy (τ_*, π_*) *is optimal for the stochastic optimal control problem* (5.14) *if and only if the function* $\Xi_{\alpha^*,\beta^*}(f)$ *is completely monotone, where, for any* a, b *with* $0 \le a \le b$, $\Xi_{a,b}(f)$ *is defined as follows:*

$$\Xi_{a,b}(f) : \theta \mapsto -\frac{\mathrm{e}^{\theta b}}{\theta} \int_{(\beta,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G^{(a)}(\mathrm{d}z). \tag{5.20}$$

Remark 5.15. (i) The complete monotonicity of the function $\Xi_{a,b}(f)$ defined in Eqn. (5.20) is equivalent to the following condition:

$$ {}_0\mathcal{L}_\infty^w V_{\tau_*,\pi_*}(x) \le 0 \qquad \text{for all } x > \beta_+^*. \tag{5.21}$$

This assertion as well as the statement in Thm. 5.14(ii) follow by a line of reasoning analogous to the one employed in the proof of Thm. 5.4(ii).

(ii) The definitions of α^* and β^* imply that the function $V = V_{\tau_*,\pi_*}$ satisfies the following two inequalities:

$$\begin{cases} V(x) - V(y) \ge x - y - K & \forall x, y \ge 0,\ x \ge y, \\ V(x) \ge f(x) & \forall x \ge 0. \end{cases} \tag{5.22}$$

(iii) The assertion in Thm. 5.14(i) follows on account of the fact that $F^{(\alpha^*)}(x-\alpha^*)$ is a super-solution for (5.14) in the sense of Def. 4.4, by reasoning as in Rem. 5.6). The super-solution property is a consequence of the observation in part (ii) and the fact that

$$\mathrm{e}^{-q(t\wedge T_{\alpha^*})}F^{(\alpha_*)}(X_{t\wedge T_{\alpha^*}}-\alpha^*)$$

is a martingale.

5.4. **Recursion for the dividend band levels.** The candidate optimal levels $\underline{a}^*$, $\underline{b}^*_-$ and $\underline{b}^*_+$ can next be defined recursively by solving repeatedly the optimal control problem (5.14) for appropriate reward functions f, using the following procedure:

0. **Set** $i \leftarrow 1$, $\underline{a}^* \leftarrow \{0\}$, $\underline{b}^* \leftarrow \{b^*(0)\}$, $f \leftarrow {}_{b^*_+}v^*_b$ and let Ξ be given by eqn. (5.11).
1. **If** Ξ is completely monotone, **set** $\underline{a}^* \leftarrow \underline{a}^* \cup \{\infty\}$. **Return** $\{\underline{a}, \underline{b}\}$.
2. **Else** define
$$a_{i+1} \leftarrow \alpha^*_f \qquad \text{and} \qquad b^*_{i+1} \leftarrow \beta^*_f.$$
3. **Set** $\underline{a}^* \leftarrow \underline{a} \cup \{a^*_{i+1}\}$, $\underline{b}^* \leftarrow \underline{b} \cup \{b^*_{i+1}\}$, $f \leftarrow {}_{b^*_{i+1,+}}V_{\underline{a}^*,\underline{b}^*}$, $\Xi \leftarrow \Xi_{\underline{a}^*,\underline{b}^*}(f)$, $i \leftarrow i+1$.
Go to step 1.

Remark 5.16. (i) If, in step 1 of the ith iteration of above procedure, the function Ξ is not completely monotone, then the function $f := {}_{b^*_{i,+}}V_{\underline{a}^*,\underline{b}^*}$ satisfies the conditions in As. 1 with $w = f|_{(-\infty,0)}$. To see that this is the case, observe that As. 1(i) is satisfied on account of the form of $f|_{\mathbb{R}_+}$, that As. 1(ii) holds follows as in Rem. 5.15, and that As. 1(iii) follows in view of Prop. D.3 and the definition of the levels $\underline{a}^*, \underline{b}^*$.

(ii) In the absence of transaction cost ($K=0$) there may exist a limit point $\gamma_* = \lim_{i\to\infty} b^*_{i,+} = \lim_{i\to\infty} a^*_i$ of the band levels. In that case the procedure will converge to the value-function $V_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}$ corresponding to the levels $\underline{\tilde{a}}^* = (a^*_i)$, $\underline{\tilde{b}}^* = (b^*_i)$, and needs to be re-started as follows:

0.′ **Set** $i \leftarrow 1$, $\underline{a}^* \leftarrow \underline{\tilde{a}}^*$, $\underline{b}^* \leftarrow \underline{\tilde{a}}^*$, $f \leftarrow {}_{\gamma^*}V_{\underline{\tilde{a}}^*,\underline{\tilde{a}}^*}$, $\Xi \leftarrow \Xi_{\underline{\tilde{a}}^*,\underline{\tilde{a}}^*}(f)$.

Denote by $\underline{v}^* = (v_i, i \in \mathcal{I})$, $\underline{a}^* = (a_i, i \in \mathcal{I})$ and $\underline{b}^* = (b^*_i, i \in \mathcal{I})$ the sequence of value-functions and band levels generated by above procedure (possibly re-started), where $\mathcal{I}$ is a corresponding index set. The solution of the stochastic optimal control problem (3.2) is expressed in terms of the sequences $\underline{v}$, $\underline{a}$ and $\underline{b}$ in the following result, which in particular implies Thm. 3.3:

Proposition 5.17. *(i) For each $i \in \mathcal{I}$, v_i is equal to the value-function $v_{\underline{a}_i,\underline{b}_i}$ of the multi-bands strategy $\pi_{\underline{a}_i,\underline{b}_i}$ at levels $\underline{a}_i = (0, a^*_1, \ldots, a^*_i)$ and $\underline{b}_i = (b^*_1, \ldots, b^*_i)$.*

(ii) For each $i \in \mathcal{I}$, $v_i(x) = v_*(x)$ for all $x \le b^*_{i,+}$.

(iii) The value function $v_{\underline{a}^*,\underline{b}^*}$ of the strategy $\pi_{\underline{a}^*,\underline{b}^*}$ is equal to v^*.

Proof. (i) For each $i > 1$, the strong Markov property applied at $\sigma_{a^*_{i-1}}$ implies that $v_{\pi_{\underline{a}^*_i,\underline{b}^*_i}}(x)$ is equal to the value $V_{a^*_i,b^*_i}(x)$ of the strategy $(\sigma_{a^*_{i-1}}, \pi_{b^*_i})$ with $f_w(x) = v_{\pi_{\underline{a}^*_{i-1},\underline{b}^*_{i-1}}}(b^*_{i-1}+x)$. For $i=1$ a similar

observation applies with reward function $f(x) = x + \gamma_w$. In view of the equality $V_{a_i^*,b_i^*} = v_{\underline{a}_i^*,\underline{b}_i^*}$ then follows by induction.

Statement (ii) follows in view of (i), following an analogous line reasoning to the one employed in Rem. 5.15(iii).

(iii) Note that the sequence $(a_i, i \in \mathcal{I})$ is strictly increasing and ultimately tends to infinity (cf. Step 1 of the above procedure and Lem. 5.13) and that $v_i(x) = v_{\underline{a}^*,\underline{b}^*}(x)$ for all $x \leq a_i^*$. Hence, on account of (i) and (ii), it follows that $v_*(x) = v_{\underline{a}^*,\underline{b}^*}(x)$, for any fixed $x \in \mathbb{R}_+$. □

Remark 5.18. (i) When either there are no transaction cost ($K = 0$) or $K > 0$ and ν has finite mass, the optimal value function is affine for sufficiently large levels of the reserves (Prop. 4.13). Hence, in this case, there will exist a finite highest band of the multi-band strategy, with upper level given as follows:

$$y^* = \inf\{x : v_*(y) = y - x + v_*(x)\ \forall y \geq x\}.$$

(ii) In Shreve *et al.* (1994) an explicit example is given of an optimal control problem in a diffusion setting in which a multi-dividend-bands strategy is optimal with countably many bands. The construction of an explicit example in the current setting in which a multi-dividend-bands strategy with countably many bands is optimal is left as an open problem.

6. Examples

We compute first the generating scale function for processes whose homogeneous scale admits an exponential decomposition, i.e. assuming

$$W^{(q)}(x) \quad = \quad \sum A_i \mathrm{e}^{\zeta_i(q)x},$$

where $\zeta_i(q)$ are the roots of the Cramér-Lundberg equation $\psi(\zeta) = q$. This implies that

$$\begin{aligned} Z^{(q,v)}(x) &= \mathrm{e}^{vx}(1 + (q - \psi(v)) \int_0^x \mathrm{e}^{-vy} W^{(q)}(y)\mathrm{d}y) = \mathrm{e}^{vx} + (q - \psi(v)) \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x} - \mathrm{e}^{vx}}{\zeta_i(q) - v} \\ &= (\psi(v) - q) \sum_i \frac{A_i}{v - \zeta_i(q)} \mathrm{e}^{\zeta_i(q)x}, \end{aligned}$$

where we have used that $\sum \frac{A_i}{v - \zeta_i(q)} = \frac{1}{\psi(v) - q}$. In particular,

$$\begin{aligned} Z^{(q)}(x) &= q \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i(q)}, \\ Z_1(x) &= \overline{Z}^{(q)}(x) - \psi'(0)\overline{W}^{(q)}(x) = q \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i^2(q)} - \psi'(0) \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i(q)}. \end{aligned}$$

To determine the optimality of a single dividend-band strategy in the presence of an exponential penalty $w(x) = c\mathrm{e}^{vx}$ or a linear penalty $w(x) = cx$, we will study the extrema of the functions

$$G^{(v)}(x) := \frac{1 - cZ^{(q,v)\prime}(x)}{W^{(q)\prime}(x)}, \qquad G_1(x) := \frac{1 - cZ_1'(x)}{W^{(q)\prime}(x)}. \tag{6.1}$$

Theorem 5.4(ii) yields the following sufficient optimality condition in terms of $G^{(v)}$ and G_1:

Lemma 6.1. *a) The unimodality of the functions $G^{(v)}$ and $G_1 : \mathbb{R}_+ \to \mathbb{R}$ defined in* (6.1) *implies the optimality of single dividend-band policies if $w(x) = c\mathrm{e}^{vx}$, $c < 0$, and $w(x) = cx$, $c > 0$, respectively.*

b) In particular, if $K = 0$ and $G^{(v)}$ and G_1 are monotone decreasing, then a "liquidation" strategy is optimal.

Of interest is therefore the sign of the functions $D^{(v)}(x) = -G^{(v)\prime}(x)W^{(q)\prime}(x)^2$ and $D_1(x) = -G_1'(x)W^{(q)\prime}(x)^2$, which are explicitly given by the following expressions:

$$\begin{aligned}
D^{(v)}(x) &= W^{(q)\prime\prime}(x)(1 - cZ^{(q,v)\prime}(x)) + cZ^{(q,v)\prime\prime}(x)W^{(q)\prime}(x) \\
&= \sum_j A_j\zeta_j(q)^2\mathrm{e}^{\zeta_j(q)x} - c(\psi(v) - q)\sum_j\sum_{k>j}\frac{\zeta_j(q)\zeta_k(q)(\zeta_j(q) - \zeta_k(q))^2}{(v - \zeta_j(q))(v - \zeta_k(q))}A_jA_k\mathrm{e}^{(\zeta_j(q)+\zeta_k(q))x}, \\
D_1(x) &= \sum_j A_j\zeta_j(q)^2\mathrm{e}^{\zeta_j(q)x} - c\sum_j\sum_{k>j}\left(\psi'(0) - q\frac{(\zeta_j(q) + \zeta_k(q))}{\zeta_j(q)\zeta_k(q)}\right)(\zeta_j(q) - \zeta_k(q))^2A_jA_k\mathrm{e}^{(\zeta_j(q)+\zeta_k(q))x}.
\end{aligned}$$

6.1. **Cramér-Lundberg model with exponential jumps.** Suppose next that X is given by the Cramér-Lundberg model (1.1) with exponential jump sizes with mean $1/\mu$, jump rate λ, and Laplace exponent $\psi(\theta) = p\theta - \lambda\theta/(\mu + \theta)$. The homogeneous scale function is:

$$W^{(q)}(x) \;=\; A_+\mathrm{e}^{\zeta^+(q)x} - A_-\mathrm{e}^{\zeta^-(q)x},$$

where $A_\pm = p^{-1}\frac{\mu+\zeta^\pm(q)}{\zeta^+(q)-\zeta^-(q)}$, and $\zeta^+(q) = \Phi(q)$, $\zeta^-(q)$ are the largest and smallest roots of the polynomial $(\psi(\theta) - q)(\theta + \mu)$:

$$\zeta^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}.$$

Hence, in this case we find

$$\begin{aligned}
Z^{(q)}(x) &= q\left(\frac{A_+}{\zeta^+(q)}\mathrm{e}^{\zeta^+(q)x} - \frac{A_-}{\zeta^-(q)}\mathrm{e}^{\zeta^-(q)x}\right) = \mu^{-1}\left(\zeta^+(q)A_-\mathrm{e}^{\zeta^-(q)x} - \zeta^-(q)A_+\mathrm{e}^{\zeta^+(q)x}\right) \\
&= \frac{(q - \zeta^-(q))\mathrm{e}^{\zeta^+(q)x} + (\zeta^+(q) - q)\mathrm{e}^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)}, \qquad (6.2) \\
Z^{(q,v)}(x) &= Z^{(q)}(x) + \lambda\frac{v}{v + \mu}\frac{\mathrm{e}^{\zeta^+(q)x} - \mathrm{e}^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)} = B_+\mathrm{e}^{\zeta^+(q)x} + B_-\mathrm{e}^{\zeta^-(q)x}, \\
D^{(v)}(x) &= \alpha_+\mathrm{e}^{\zeta^+(q)x} - \alpha_-\mathrm{e}^{\zeta^-(q)x} + c\alpha_o\mathrm{e}^{(\zeta^+(q)+\zeta^-(q))x}, \qquad (6.3)
\end{aligned}$$

where $B_{\pm} = \pm q\frac{A_{\pm}}{\zeta^{\pm}(q)} \pm \frac{\lambda v}{v+\mu}\frac{1}{\zeta^{+}(q)-\zeta^{-}(q)}$, and

$$\alpha_{+} = A_{+}(\zeta_{+}(q))^2 > 0, \qquad \alpha_{-} = A_{-}(\zeta_{-}(q))^2 > 0, \qquad \alpha_o = +\frac{q\mu}{p^2}\frac{C}{v+\mu} > 0, \tag{6.4}$$

$$\text{where} \qquad C = (\mu+\zeta_{+}(q))(\mu+\zeta_{-}(q)) = \frac{\lambda\mu}{p} > 0. \tag{6.5}$$

In particular, using that $(\zeta^{+}(q) + \zeta^{-}(q))/(\zeta^{+}(q)\,\zeta^{-}(q)) = \psi'(0)/q - 1/\mu$, yields

$$\begin{aligned} Z_1(x) &= \lambda\mu^{-1}\frac{\mathrm{e}^{\zeta^{+}(q)x} - \mathrm{e}^{\zeta^{-}(q)x}}{\zeta^{+}(q)-\zeta^{-}(q)} = C_{+}\mathrm{e}^{\zeta^{+}(q)x} + C_{-}\mathrm{e}^{\zeta^{-}(q)x}, \\ D_1(x) &= \alpha_{+}\mathrm{e}^{\zeta^{+}(q)x} - \alpha_{-}\mathrm{e}^{\zeta^{-}(q)x} - c\,\omega_o\mathrm{e}^{(\zeta^{+}(q)+\zeta^{-}(q))x}, \end{aligned} \tag{6.6}$$

where $C_{\pm} = \pm\lambda\mu^{-1}(\zeta^{+}(q)-\zeta^{-}(q))^{-1}$ and $\omega_o = \frac{q}{\mu p^2}C$.

Let us recall next that in the absence of penalty and costs ($w(x) = K = 0$), the function $G(x)^{-1} = W^{(q)\prime}(x)$ is unimodal [8] with global minimum at

$$b^* = \frac{1}{\zeta^{+}(q)-\zeta^{-}(q)}\begin{cases}\log\frac{\zeta^{-}(q)^2(\mu+\zeta^{-}(q))}{\zeta^{+}(q)^2(\mu+\zeta^{+}(q))} & \text{if } W^{(q)\prime\prime}(0) < 0 \Leftrightarrow (q+\lambda)^2 - p\lambda\mu < 0\\ 0 & \text{if } W^{(q)\prime\prime}(0) \geq 0 \Leftrightarrow (q+\lambda)^2 - p\lambda\mu \geq 0\end{cases}$$

(since $W^{(q)\prime\prime}(0) \sim \zeta^{+}(q)^2(\mu+\zeta^{+}(q)) - \zeta^{-}(q)^2(\mu+\zeta^{-}(q))/(\zeta^{+}(q)) - \zeta^{-}(q)) = (q+\lambda)^2 - p\lambda\mu$ and therefore the optimal strategy is always the barrier strategy at level b^*. The case $K = 0, w(x) = cx$ was tackled in [9].

More generally, when w is exponential or linear and $K \geq 0$ the function G is unimodal, which is a consequence of the following auxiliary result:

Lemma 6.2. *Let $\alpha_i, \lambda_i \in \mathbb{R}$, $i = 1,2,3$ such that $\alpha_1 > 0 > \alpha_3$, and $\lambda_1 > \lambda_2 > \lambda_3$. Then the function $f(x) := \sum_{i=1}^{3}\alpha_i\mathrm{e}^{\lambda_i x}$ has a unique root c^* of $f(c^*) = 0$, which satisfies $f'(c^*) > 0$, and it holds that*

$$f(x) > 0 \qquad \text{for all } x > c^*.$$

In particular, let $h : \mathbb{R}_+ \to \mathbb{R}$ and $k : \mathbb{R}_+ \to (0,\infty)$ be such that $h'(x) = k(x)f(x)$ for $x > 0$, then h is unimodal.

Proof. The function $g(x) := \mathrm{e}^{-\lambda_3 x}f(x)$ tends to $+\infty$ and to $\alpha_3 < 0$ as $x \to \pm\infty$. If $\alpha_2 \geq 0$, g is strictly convex and strictly increasing. If $\alpha_2 < 0$, g attains a minimum at the unique root of g'. In both cases $g(c) = 0$ admits a unique root c, and it holds that $g'(c) > 0$. As a consequence c is a unique root of $f(x) = 0$, $f'(c) > 0$, and $f(x) > 0$ for $x > c$. In particular, h has a unique stationary point where it attains a maximum, so that it is unimodal. □

The form of $D^{(v)}(x)$ and $D_1(x)$, Lem. 6.2 and the fact that $W^{(q)\prime}(x) > 0$ imply that $G^{(v)}$ and G_1 are unimodal. Hence, single barrier policies are optimal, in view of Lem. 6.1. Let us next characterize the optimal level b^*.

For $K \geq 0$, we find:

(1) Let $K = 0$. In the case of an exponential penalty, $b_+^* = 0$ iff

$$G^{(v)\prime}(0) \leq 0 \Leftrightarrow (q+\lambda)^2 - \lambda\mu p \geq -c\lambda q \frac{\mu^2}{v+\mu},$$

as follows from (6.3). Similarly, in the case of linear penalty, it holds that $b_{1,+}^* = 0$ iff

$$G_1'(0) \leq 0 \Leftrightarrow (q+\lambda)^2 - \lambda\mu p \geq c\lambda q,$$

in view of (6.6). If b_+^* is positive, it is a stationary point, and hence solves the equation

$$G^{(v)\prime}(b) = 0 \Leftrightarrow 0 = D^{(v)}(b) = \alpha_+ \mathrm{e}^{\zeta^+(q)b} - \alpha_- \mathrm{e}^{\zeta^-(q)b} + c\alpha_o \mathrm{e}^{(\zeta^+(q)+\zeta^-(q))b}, \tag{6.7}$$

if the penalty w is exponential and

$$G_1'(b) = 0 \Leftrightarrow 0 = D_1(b) = \alpha_+ \mathrm{e}^{\zeta^+(q)b} - \alpha_- \mathrm{e}^{\zeta^-(q)b} - c\beta_o \mathrm{e}^{(\zeta^+(q)+\zeta^-(q))b}, \tag{6.8}$$

if w is a linear penalty.

(2) Suppose next that $K > 0$. Then b_+^* is strictly positive as a consequence of the positive transaction cost K, and the optimal levels (b_-^*, b_+^*) are given by $(b_-^*, b_*^+ + d^*)$ where (b,d) maximizes over $(b,d) \in \mathbb{R}_+ \times (0,\infty)$ the function

$$(b,d) \mapsto \frac{d - K - B_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d} - 1) + B_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d} - 1)}{A_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d} - 1) - A_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d} - 1)} \tag{6.9}$$

if w is an exponential penalty, and the function

$$(b,d) \mapsto \frac{d - K - C_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d} - 1) + C_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d} - 1)}{A_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d} - 1) - A_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d} - 1)} \tag{6.10}$$

if w is a linear penalty.

The following result sums up the form of the optimal dividend policy:

Lemma 6.3. *Consider a Cramér-Lundberg process (1.1) with exponential jump sizes with mean $1/\mu$, and fixed cost $K \geq 0$. The optimal dividend policy is given by a* single dividend-band strategy π_{b^*} *for the following Gerber-Shiu penalties w:*

a) Exponential penalties: $w(x) = c\mathrm{e}^{xv}$, $c < 0$.

- (i) *If $K = 0$ and $(q+\lambda)^2 - \lambda\mu p \geq -c\lambda q \frac{\mu^2}{v+\mu}$, then $b^* = 0$.*
- (ii) *If $K = 0$ and $(q+\lambda)^2 - \lambda\mu p < -c\lambda q \frac{\mu^2}{v+\mu}$, then b^* is the unique solution $b \in (0,\infty)$ of* (6.7).
- (iii) *If $K > 0$, $b_+^* = b_-^* + d^*$ where b_-^* and d^* maximize over $b \geq 0$, $d > 0$, the function* (6.9).

b) Linear penalties: $w(x) = cx$, $c \geq 0$.

- (i) *If $K = 0$ and $(q+\lambda)^2 - \lambda\mu p \geq c\lambda q$, then $b^* = 0$.*
- (ii) *If $K = 0$ and $(q+\lambda)^2 - \lambda\mu p < c\lambda q$, then b^* is the unique solution $b \in (0,\infty)$ of* (6.8).
- (iii) *If $K > 0$, $b_+^* = b_-^* + d^*$ where $b_{1,-}^* \geq 0$ and $d^* > 0$ maximize over (b,d), the function* (6.10).

Appendix A. Properties of F_w and $Z^{(q,v)}$

For reference, we list some properties of the function F_w. Recall the definition of the set $\mathcal{R}$ was given in Def. 2.7.

Lemma A.1. *Let $w \in \mathcal{R}$. Then the following hold true:*

(i) The function F_w can be expressed in terms of $W^{(q)}$ as follows:

$$F_w(x) = \frac{\sigma^2}{2} w'(0-)W^{(q)}(x) + w(0)Z^{(q)}(x) - \int_0^x W^{(q)}(x-y)w_\nu(y)\mathrm{d}y, \quad x \geq 0. \tag{A.1}$$

(ii) The value of F_w at $x = 0$ matches $w(0)$: $F_w(0) = w(0)$.

(iii) The following asymptotics hold true:

$$\frac{F_w(x)}{W^{(q)}(x)} \sim \kappa_w, \qquad \text{as } x \to \infty,$$

where κ_w is defined in Eqn. (2.26).

(iv) If X has paths of bounded variation, the Laplace transform of F_w simplifies as follows:

$$\int_0^\infty \mathrm{e}^{-\theta x} F_w(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}[pw(0) - \widetilde{w}^*_\nu(\theta)], \tag{A.2}$$

where $\widetilde{w}^*_\nu$ is the Laplace transform of $\widetilde{w}_\nu : (0,\infty) \to \mathbb{R}$ given by $\widetilde{w}_\nu(x) = \int_{(x,\infty)} w(x-y)\nu(\mathrm{d}y)$.

Note that representation (A.1) and the continuity of $W^{(q)}|_{\mathbb{R}_+}$ imply that the Laplace-transform (2.24) admits a continuous version on $\mathbb{R}_+$, which justifies Def. 2.8.

Proof: (i) The identity follows by term-wise inverting the Laplace transform (2.24), using the form (1.4) of the Laplace transform of $W^{(q)}$.

(ii) This follows directly from Eqn. (A.1) and the facts that $Z^{(q)}(0) = 1$ and $\sigma^2 W^{(q)}(0) = 0$.

(iii) Since $W^{(q)}(x) \sim \mathrm{e}^{\Phi(q)x}/\psi'(\Phi(q))$ as $x \to \infty$, the statement follows from Eqn. (A.1).

(iv) If X has bounded variation then $\overline{\nu}_1 := \int_0^\infty \overline{\nu}(x)\mathrm{d}x < \infty$. Hence, for any $x > 0$, $\widetilde{w}_\nu$ is finite and equal to $\widetilde{w}_\nu(x) = w_\nu(x) - w(0)\overline{\nu}_1$. □

Restricting to penalties w from the set P, which was defined in Def. 3.1, we have the following additional properties:

Lemma A.2. *Let $w \in \mathcal{P}$.*

(i) The function $w_\nu : (0,\infty) \to \mathbb{R}$ defined in Eqn. (2.23) is increasing and right-continuous, and satisfies the following integrability condition:

$$\int_0^x |w_\nu(y)|\mathrm{d}y < \infty \qquad \text{for any } x > 0. \tag{A.3}$$

(ii) The function $J_w : (0,\infty) \to \mathbb{R}$ given by $J_w(x) = {}_0\mathcal{L}^w_\infty p_w(x)$ with $p_w(x) = w'(0-)x + w(0)$ is right-continuous, and is equal to the following expression:

$$J_w(y) = [\psi'(0) - m_\nu(y)]w'(0-) + w_\nu(y) - q(w'(0-)y + w(0)), \qquad y > 0, \tag{A.4}$$

where w_ν *is given in* (2.23) *and the map* $m_\nu : (0,\infty) \to (-\infty, 0)$ *is given by* $m_\nu(x) = \int_x^\infty (x-z)\nu(\mathrm{d}z)$.

(iii) $F_w(x)$ *is left- and right-differentiable at any* $x > 0$ *with right-derivative at* $x > 0$ *given by*

$$F_w'(x) = \frac{\sigma^2}{2} w'(0-) W^{(q)\prime}(x) + w(0) q W^{(q)}(x) - \int_0^x w_\nu(x-y) W^{(q)}(\mathrm{d}y), \tag{A.5}$$

where the first term is zero if $\sigma^2 = 0$. *If* $\sigma^2 > 0$ *or* $\nu_1 = \infty$, *then* $F_w|_{(0,\infty)} \in C^1(0,\infty)$.

(iv) The following alternative representation of $F_w'(x)$ *holds true:*

$$F_w'(x) = w'(0-) - \int_0^x J_w(x-y) W^{(q)}(\mathrm{d}y), \qquad x > 0. \tag{A.6}$$

(v) The right-derivative at $x = 0$ *of* F_w *takes the following form:*

$$F_w'(0+) = \begin{cases} w'(0-), & \text{if } \sigma^2 > 0 \text{ or } \nu_1 = \infty, \\ -J_w(0+) = \frac{q}{p} w(0) - \frac{1}{p} w_\nu(0), & \text{if } \sigma^2 = 0 \text{ and } \nu_1 < \infty, \end{cases} \tag{A.7}$$

where $\nu_1 = \int_0^1 x\nu(\mathrm{d}x)$.

(vi) The map $F_w : (0,\infty) \to \mathbb{R}$ *is equal to a difference of monotone functions.*

Proof: (i) The integrability condition (A.3) follows from the condition (2.22) (as $\mathcal{P} \subset \mathcal{R}$, as noted just before Def. 3.1). The right-continuity and monotonicity of w_ν follow on account of the dominated convergence theorem and the monotonicity and right-continuity of w.

(ii) The representation in Eqn. (A.4) follows directly from the form of the operator ${}_0\mathcal{L}^w_\infty$ given in Eqn. (4.5). The function J_w inherits the right-continuity from w_ν, on account of in view of Eqn. (A.4) and the continuity of m_ν.

(iii) Recall that $W^{(q)}(x)$ is right- and left-differentiable at any $x > 0$ (with finite derivatives and with right-derivative at x denoted by $W^{(q)\prime}(x)$). The final term on the rhs of Eqn. (A.1) is also right-differentiable with derivative equal to the third term on the rhs pf Eqn. (A.5), on account of the dominated convergence theorem, the monotonicity and right-continuity of w_ν and the right-differentiability of $W^{(q)}$. A analogous reasoning shows that $F_w'|_{(0,\infty)}$ is in fact continuous if $\sigma^2 > 0$ or $\nu_1 = \infty$, employing the fact that in that case $W^{(q)}|_{(0,\infty)}$ is C^1.

(iv) The equality of (A.5) and (A.6) can be verified by taking Laplace transforms, using that the Laplace transforms of $W^{(q)}$ and m_ν are given by Eqn. (1.4) and by the following expression:

$$m_\nu^*(\theta) = \theta^{-2} \int_0^\infty [\mathrm{e}^{-\theta z} - 1 + \theta z]\nu(\mathrm{d}z) = \theta^{-2}\left[\psi(\theta) - \theta\psi'(0) - \tfrac{\sigma^2}{2}\theta^2\right].$$

(v) If X has bounded variation, then $w_\nu(0+)$ exist and is finite. On account of the monotonicity of w_ν, the continuity of $W^{(q)}$ and $W^{(q)}(0) = p^{-1}$, the expression in Eqn. (A.7) follows by taking the limit of x to 0 in Eqn. (A.5). If X has unbounded variation, the form of $F_w'(0+)$ follows on account of the fact that the convolution in Eqn. (A.6) vanishes as x tends to zero. This fact follows on account of the following two observations: (α) Let $\eta > 0$ and $\delta > 0$ be such that, for all $z \in (0,\delta)$,

$|\delta w(-z) - w'(0-)| \leq \eta$, where $\Delta w(z) = \frac{w(z)-w(0)}{z}$. Then the form of w_ν implies that the following estimate holds true:

$$|w_\nu(x)| \leq \int_{[\delta,\infty)} |w(-y) - w(0-)|\nu(\mathrm{d}y) + \eta|m_\nu(x)|, \qquad x > 0. \tag{A.8}$$

(β) For any $a, b \geq 0$, define the function $K : (0,\infty) \to \mathbb{R}$ by $K(x) := \int_0^x (a - bm_\nu(x-y)W^{(q)}(\mathrm{d}y)$. As K is increasing and has a Laplace transform $K^*(\theta) = (\psi(\theta)-q)^{-1}\theta(a-bm_\nu^*(\theta))$ that satisfies $K^*(\theta) \sim c/\theta$ as θ tends to infinity for some constant c, a Tauberian theorem implies that $K(x)$ tends to zero as x tends to zero. The stated fact now follows by combining the observations (α) and (β).

(vi) The statement follows on account of the representation in Eqn. (A.5) and the facts that w_ν is monotone and non-positive and that $W^{(q)\prime}|_{(0,\infty)}$ is equal to the difference of two monotone functions which holds as $W^{(q)}$ is log-concave, which is in turn a consequence of the representation of $W^{(q)}$ in terms of the excursion measure ([12, p.195]). □

In the case of exponential boundary condition w we record the following additional properties:

Remark A.3. The family of functions $Z^{(q,v)}$ contains as member the function $Z^{(q,0)} = Z^{(q)}$, which corresponds to the case of a boundary condition equal to 1. Further, from (2.17) we read off that $Z_0 = Z^{(q)}$, and if $E[|X_1|] < \infty$, that $Z_1(x)$ is given by

$$Z_1(x) = x + q\overline{W}^{(q,1)}(x) - \psi'(0)\overline{W}^{(q)}(x), \tag{A.9}$$

where $\overline{W}^{(q,1)}(x) = \int_0^x (x-y)W^{(q)}(y)\mathrm{d}y$. More generally, if $E[|X_1|^k] < \infty$, then $\psi^{(r)}(0)$ is finite for $r = 1, \ldots, k$, and the following representation holds true by an application of the Leibniz rule:

$$Z_k(x) = x^k + q\overline{W}^{(q,k)}(x) - \sum_{n=1}^{k} \binom{k}{n} \psi^{(n)}(0)\overline{W}^{(q,k-n)}(x) \tag{A.10}$$

with $\psi^{(n)}(0)$ being the nth right-derivative of ψ at zero and

$$\overline{W}^{(q,n)}(x) = \int_0^x (x-y)^n W^{(q)}(y)\mathrm{d}y.$$

Remark A.4. (i) For $v \geq 0$, the function $x \mapsto Z^{(q,v)}(x)$ is strictly increasing on $\mathbb{R}_+$. In particular, for $x > 0$ and $v > \Phi(q)$, $Z^{(q,v)\prime}(x)$ is equal to

$$Z^{(q,v)\prime}(x) = (\psi(v) - q) \int_x^\infty \mathrm{e}^{v(x-y)} W^{(q)}(\mathrm{d}y). \tag{A.11}$$

which can be derived from Eqns. (1.4) and (2.16) by integration by parts.

(ii) The map $v \mapsto Z^{(q,v)\prime}(x)$ is completely monotone on $(\Phi(q),\infty)$, for any $x > 0$. This follows since $v \mapsto Z^{(q,v)}(x)$ is the Laplace transform of some measure on $\mathbb{R}_+$. Indeed, a straightforward calculation, using the definition of $Z^{(q,v)}$ and Eqn. (1.4), shows that the Laplace transform of $Z^{(q,v)}$ admits the

following representation:

$$\int_0^\infty \mathrm{e}^{-\lambda x} Z^{(q,v)\prime}(x)\mathrm{d}x = \frac{q}{v}\cdot\frac{1}{\psi(\lambda)-q} + \frac{\lambda}{\psi(\lambda)-q}\left[\frac{\sigma^2}{2} + \int_{(0,\infty)}\int_{(0,\infty)} \mathrm{e}^{-\lambda s - vt}\overline{\nu}(s+t)\mathrm{d}t\mathrm{d}s\right].$$

Inverting this Laplace transform in λ yields the following expression for $Z^{(q,v)\prime}$ in terms of $W^{(q)}$:

$$Z^{(q,v)\prime}(x) = \frac{q}{v}W^{(q)}(x) + \frac{\sigma^2}{2}W^{(q)\prime}(x) + \int_{[0,x]}\int_{(0,\infty)} \mathrm{e}^{-vt}\overline{\nu}(x-y+t)\mathrm{d}t\, W^{(q)}(\mathrm{d}y), \qquad x>0.$$

For any $x\in\mathbb{R}_+$ the function $v\mapsto Z^{(q,v)\prime}$ is the Laplace transform of a measure on $\mathbb{R}_+$ which thus implies the stated complete monotonicity.

(iii) If, for some $v_0>0$, $E[\mathrm{e}^{-v_0X_1}]$ is finite, $\psi(v)$ and $v\mapsto Z^{(q,v)}(x)$ can be analytically extended into a neighbourhood of $v=0$, and $Z^{(q,v)}(x)$ can be expanded in terms of Z_k, $k\in\mathbb{N}$, as follows:

$$Z^{(q,v)}(x) = \sum_{k=0}^\infty \frac{v^k}{k!}Z_k(x). \tag{A.12}$$

Remark A.5 (Proof of Prop. 2.5)**.** Note that, by changing measure and inserting the identity Eqn. (2.12), the following expression can be derived for $v\geq 0$:

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-qT_{a,b}+v(X_{T_{a,b}}-a)}\mathbf{1}_{\{T_a^-<T_b^+\}}]\\
&= \mathrm{e}^{(x-a)v}\mathbb{E}_x[\mathrm{e}^{-qT_{a,b}+\psi(v)T_{a,b}+v(X_{T_{a,b}}-x)-\psi(v)T_{0,a}}\mathbf{1}_{\{T_a^-<T_b^+\}}]\\
&= \mathrm{e}^{(x-a)v}\mathbb{E}_x^v[\mathrm{e}^{-(q-\psi(v))T_{a,b}}\mathbf{1}_{\{T_a^-<T_b^+\}}]\\
&= \mathrm{e}^{(x-a)v}\left[Z_v^{(q-\psi(v))}(x-a) - \frac{Z_v^{(q-\psi(v))}(b-a)}{W_v^{(q-\psi(v))}(b-a)}W_v^{(q-\psi(v))}(x-a)\right],
\end{aligned}\tag{A.13}$$

where $W_v^{(r)}, Z_v^{(r)}$ are the r-scale functions under $\mathbb{P}^v$, the Cramér-Esscher change of measure of $\mathbb{P}$ with Radon-Nikodym derivative defined by $\frac{\mathrm{d}\mathbb{P}^v}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \exp(vX_t-\psi(v)t)$. Using the identity (from [7])

$$W^{(q)}(x) = \mathrm{e}^{xv}W_v^{(q-\psi(v))}(x), \qquad v\geq 0, q\geq 0,$$

we find (2.19). The identity (2.20) follows by a similar line of reasoning, starting from (2.13). The uniqueness follows from Thm. 2.12. □

Remark A.6 (Proof of Prop. 2.10)**.** Writing $\mathcal{V}_w^{0,\infty}(x) = w(0)\mathbb{E}_x[\mathrm{e}^{-qT_0^-}] + \mathbb{E}_x[\mathrm{e}^{-qT_0^-}(w(X_{T_0^-})-w(0))]$ and applying the compensation formula to the Poisson point process $(\Delta X_t, t\in\mathbb{R}_+)$ yield the following expressions for any $x\in\mathbb{R}_+$:

$$\mathcal{V}_w^{0,\infty}(x) - w(0)\mathcal{V}_{e_0}^{0,\infty}(x) = \int_0^\infty\int_y^\infty (w(y-z)-w(0))\nu(\mathrm{d}z)U^q(x,\mathrm{d}y) \tag{A.14}$$

$$= W^{(q)}(x)w_\nu^*(\Phi(q)) - \int_0^x W^{(q)}(x-y)w_\nu(y)\mathrm{d}y, \tag{A.15}$$

where $U^q(x, \mathrm{d}y)$ is the q-potential measure of X under $\mathbb{P}_x$ killed upon entering $(-\infty, 0)$,

$$U^q(x, \mathrm{d}y) = [W^{(q)}(x)\mathrm{e}^{-\Phi(q)y} - W^{(q)}(x-y)]\mathrm{d}y, \qquad y > 0,$$

and $\mathcal{V}^{0,\infty}_{e_0}(x) = \mathbb{E}_x[\mathrm{e}^{-qT_0^-}]$ is expressed in terms of the scale function $W^{(q)}$ by

$$\mathcal{V}^{0,\infty}_{e_0}(x) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x).$$

The two integrals in (A.15) are finite in view of the integrability condition (2.22) and the fact that $W^{(q)}|_{\mathbb{R}_+}$ is continuous. Thus, Eqn. (2.25) follows from Eqn. (A.1) (since the term $\frac{\sigma^2}{2}w'(0-)W^{(q)}(x)$ cancels).

The martingale property in Eqn. (2.27) follows from Eqn. (2.25) and the strong Markov property of X, and the fact that

$$\left(\mathrm{e}^{-q(t\wedge T_a^-)}W^{(q)}(X_{t\wedge T_a^-} - a), t \in \mathbb{R}_+\right) \qquad \text{is a } \mathbb{P}_x\text{-martingale for any } x \in \mathbb{R}. \qquad \square$$

Appendix B. Estimates for the optimal value function v_*

Proof of Lem. 4.8: (i) Let $x > y$. Denote by $\pi_\epsilon(y)$ an ϵ-optimal strategy for the case $U_0 = y$. Then a possible strategy is to immediately pay out $x - y$ and subsequently to adopt the strategy $\pi_\epsilon(y)$, so that the following holds:

$$v_*(x) \geq x - y - K + v_{\pi_\epsilon}(y) \geq v_*(y) - \epsilon + x - y - K.$$

Since this inequality holds for any $\epsilon > 0$, the lower bound in Eqn. (4.12) follows. To prove the other bound, let $\tilde{\pi}_\epsilon(x)$ denote an ϵ-optimal strategy for the case $U_0 = x$. Then a possible strategy is to not pay any dividends until the first time that the reserves hit the level x, and to subsequently follow the policy $\tilde{\pi}_\epsilon$. Hence the following bound holds:

$$v_*(y) \geq \frac{W^{(q)}(y)}{W^{(q)}(x)}(v_{\tilde{\pi}_\epsilon}(x) - F_w(x)) + F_w(y). \tag{B.1}$$

Rearranging and letting ϵ tend to zero yields the upper-bound in Eqn. (4.12). The bounds in Eqn. (4.12) and the continuity of $W^{(q)}|_{\mathbb{R}_+}$ and of $F_w|_{\mathbb{R}_+}$ directly imply that $v_*|_{\mathbb{R}_+}$ is USRC and is moreover continuous in the case $K = 0$.

(ii) If $K = 0$, integration by parts, the non-negativity of w and the condition (1.6) of "no exogeneous ruin" imply that

$$\int_0^{\tau^\pi} \mathrm{e}^{-qt}\mathrm{d}D^\pi_t = \int_0^{\tau^\pi} q\mathrm{e}^{-qs}D^\pi_s\mathrm{d}s + \mathrm{e}^{-q\tau^\pi}D^\pi_{\tau^\pi} \leq \int_0^{\tau^\pi} q\mathrm{e}^{-qs}X_s\mathrm{d}s + \mathrm{e}^{-q\tau^\pi}X_{\tau^\pi-},$$

so that

$$v_\pi(x) \leq \mathbb{E}_x\left[\int_0^\infty q\mathrm{e}^{-qs}\overline{X}_s\mathrm{d}s\right] = x + \frac{1}{\Phi(q)},$$

where we used that the running supremum $\overline{X}_{\eta(q)}$ at an independent exponential random time with mean q^{-1} follows an exponential distribution with parameter $\Phi(q)$ (e.g. [12, Cor. VII.2]). If $K > 0$,

then the above bound remains valid since the value $v_*(x)$ decreases if the transaction cost K increases. □

Proof of Lem. 4.9: The following bounds hold true:

$$\sup_{t\in\mathbb{R}_+} \mathrm{e}^{-qt} U_t^\pi \mathbf{1}_{\{t<\tau^\pi\}} \le \sup_{t\in\mathbb{R}_+} \mathrm{e}^{-qt} X_t \le \sup_{t\in\mathbb{R}_+} \int_t^\infty \mathrm{e}^{-qs}\overline{X}_s \mathrm{d}s. \tag{B.2}$$

Therefore the expectation under $\mathbb{P}_x$ of the expression on the rhs of Eqn. (B.2) is also bounded by $x + 1/\Phi(q)$.

The compensation formula applied to the Poisson point process $(\Delta X_t, t \in \mathbb{R}_+)$ and the monotonicity of w and the fact that $w(0)$ is non-positive yield that the following inequalities holds true, for any $x \in \mathbb{R}_+$:

$$\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-q\tau^\pi} w(U^\pi_{\tau^\pi})\right] &\ge w(-1) + \mathbb{E}_x\left[\mathrm{e}^{-q\tau^\pi} w(U^\pi_{\tau^\pi})\mathbf{1}_{\{U^\pi_{\tau^\pi}<-1\}}\right]\\
&= w(-1) + \int_0^\infty\int_0^\infty w(y-z)\mathbf{1}_{\{y-z<-1\}}\nu(\mathrm{d}z)\tilde{R}^q_x(\mathrm{d}y)\\
&\ge w(-1) + \int_1^\infty\int_0^\infty w(-z)\tilde{R}^q_x(\mathrm{d}y)\nu(\mathrm{d}z) \ge w(-1) + \frac{1}{q}\int_1^\infty w(-z)\nu(\mathrm{d}z),
\end{aligned} \tag{B.3}$$

where $\tilde{R}^q_x(\mathrm{d}y)$ denote the q-potential measure of U^π under $\mathbb{P}_x$,

$$\tilde{R}^q_x(\mathrm{d}y) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U^\pi_t \in \mathrm{d}y, t < \tau^\pi).$$

The rhs of (B.3) is bounded below, since the bound in Eqn. (3.1) holds as w is element of $\mathcal{P}$.

The uniform integrability of V^π follows on account of the fact that V^π is dominated by an integrable function in view of Eqn. (4.14) and Lem. 4.8(ii). □

Appendix C. Proofs of Lemmas 4.7, 4.12 and 5.13

C.1. **Proof of Lem. 4.7.** Fix arbitrary $\pi \in \Pi$, $x \in \mathbb{R}_+$ and $s, t \in \mathbb{R}_+$ with $s < t$. It is clear that V^π_t is $\mathcal{F}_t$-measurable and integrable on account of the bound in Lem. 4.8. Denote by $\Pi_s \subset \Pi$ the following set of strategies:

$$\Pi_s = \left\{\tilde{\pi} = (\pi, \overline{\pi}) = \{D^{\pi,\overline{\pi}}_u, u \ge 0\} : \overline{\pi} \in \Pi\right\}, \qquad D^{\pi,\overline{\pi}}_u = \begin{cases} D^\pi_u, & u \in [0, s);\\ D^\pi_s + D^{\overline{\pi}}_{u-s}(U^\pi_s), & u \ge s,\end{cases}$$

where $D^{\overline{\pi}}(x)$ is the strategy $\overline{\pi}$ corresponding to initial capital $X_0 = x$. Define by $W^\pi = \{W^\pi_u, u \ge 0\}$ the following value-process:

$$W^\pi_s = \operatorname*{ess.sup}_{\tilde{\pi}\in\Pi_s} J^{\tilde{\pi}}_s, \quad J^{\tilde{\pi}}_s = \mathbb{E}\left[\int_0^{\tau^{\tilde{\pi}}} \mathrm{e}^{-qu}\mu^{\tilde{\pi}}_K(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\tilde{\pi}}} w(U^{\tilde{\pi}}_{\tau^{\tilde{\pi}}})\middle|\mathcal{F}_s\right].$$

It follows that V^π is a super-martingale as direct consequence of the following $\mathbb{P}$-a.s. relations:

(i) $V^\pi_s = W^\pi_s$,

(ii) $W^\pi_s \geq \mathbb{E}[W^\pi_t|\mathcal{F}_s]$.

Proof of (ii): This identity follows by classical arguments. Since the family of random variables $\{J^{\tilde\pi}_t, \tilde\pi \in \Pi_t\}$ is directed upwards, it follows from Neveu [36] that there exists a sequence $\pi_n \in \Pi_t$ such that $J^{\tilde\pi_n}_t \uparrow W^\pi_t$. Since $\Pi_t \subset \Pi_s$ it follows that W^π_s dominates $J^{\pi_n}_s = \mathbb{E}[J^{\pi_n}_t|\mathcal{F}_s]$, so that monotone convergence implies that the following holds true:

$$W^\pi_s \geq \lim_n \mathbb{E}[J^{\pi_n}_t|\mathcal{F}_s] = \mathbb{E}[W^\pi_t|\mathcal{F}_s].$$

Proof of (i): The form of $D^{\tilde\pi}$ implies that, conditional on U^π_s, $\{D^{\tilde\pi}_u - D^{\tilde\pi}_s, u \geq s\}$ is independent of $\mathcal{F}_s$. On account of the Markov property of X it also follows that conditional on U^π_s, $\{U^{\tilde\pi}_u - U^{\tilde\pi}_s, u \geq s\}$ is independent of $\mathcal{F}_s$. As a consequence, we have the following identity on the set $\{s < \tau^\pi\}$

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau^{\tilde\pi}} \mathrm{e}^{-qu}\mu^{\tilde\pi}_K(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\tilde\pi}} w(U^{\tilde\pi}_{\tau^{\tilde\pi}})\middle|\mathcal{F}_s\right] &= \mathrm{e}^{-qs}\mathbb{E}_{U^\pi_s}\left[\int_0^{\tau^{\overline\pi}} \mathrm{e}^{-qu}\mu^{\overline\pi}_K(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\overline\pi}} w(U^{\overline\pi}_{\tau^{\overline\pi}})\right] \\ &\quad + \int_0^s \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u) \\ &= \mathrm{e}^{-qs} v_{\overline\pi}(U^\pi_s) + \int_0^s \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u).\end{aligned}$$

In particular, $\mathbb{P}_x$-a.s. the following representation holds true:

$$J^{\tilde\pi}_s = \mathrm{e}^{-q(s\wedge\tau^\pi)} v_{\overline\pi}(U^\pi_{s\wedge\tau^\pi}) + \int_0^{s\wedge\tau^\pi} \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u),$$

which yields the following $\mathbb{P}_x$-a.s. representation for W^π_s:

$$W^\pi_s = \int_0^{s\wedge\tau^\pi} \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u) + \mathrm{e}^{-q(s\wedge\tau^\pi)} \operatorname*{ess.\,sup}_{\tilde\pi=(\pi,\overline\pi)\in\Pi_s} v_{\overline\pi}(U^\pi_{s\wedge\tau^\pi}). \tag{C.1}$$

In view of the definitions of Π_s and v_*, the essential supremum in Eqn. (C.1) is $\mathbb{P}$-a.s. equal to $v_*(U^\pi_{s\wedge\tau^\pi})$, which implies that, $\mathbb{P}$-a.s., $W^\pi_s = V^\pi_s$. □

C.2. **Proof of Lem. 4.12.** Write $M^{(1)} = S^{(1)} + S^{(2)}$ where

$$S^{(1)}_t = \sum_{i\geq 1} \mathrm{e}^{-q(t\wedge T_{2i-1})}[f(X_{t\wedge T_{2i}}) - f(X_{t\wedge T_{2i-1}})], \quad S^{(2)}_t = \sum_{i\geq 1} f(X_{t\wedge T_{2i}})[\mathrm{e}^{-q(t\wedge T_{2i})} - \mathrm{e}^{-q(t\wedge T_{2i-1})}].$$

In view of the fact that $f(x) \leq ax + b$ for some constants $a, b > 0$ it follows that the following estimate holds for fixed $t > 0$:

$$|S^{(2)}_t| \leq (a\overline{X}_{t\wedge\tau_\pi} + b)\int_0^{t\wedge\tau_\pi} \mathrm{e}^{-qs} 1_{\{X_s\in(a-2\epsilon,a+2\epsilon)\}}\mathrm{d}s.$$

On account of the fact that the potential measure of X is absolutely continuous, the left-hand side tends to zero as $\epsilon \searrow 0$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}_+$. The dominated convergence theorem implies that this convergence also holds in $\mathbb{P}_x$-expectation.

For the term $S^{(1)}$ the strong Markov property applied at T_{2i-1} and Def. 2.3(i) imply that following identity holds true:

$$\mathbb{E}_x[S_t^{(1)}] = \mathbb{E}_x\left[\sum_{i\geq 1} \mathrm{e}^{-q(t\wedge T_{2i-1})} L(X_{t\wedge T_{2i-1}} - a - 2\epsilon)\right], \tag{C.2}$$

where $L(x) = F(x) - \tilde{f}(x) + \frac{W^{(q)}(x)}{W^{(q)}(4\epsilon)}(\tilde{f}(4\epsilon) - F(4\epsilon))$ with $\tilde{f} = {}_{a-2\epsilon}f$ and $F = F_{\tilde{f}}$ denotes the Gerber-Shiu function corresponding to payoff $\tilde{f}$. The triangle inequality, continuous differentiability of $\tilde{f}$ and F and the fact that $W^{(q)}$ is increasing yield the following estimate:

$$|L(x)| \leq 4\epsilon \times 2C(\epsilon) \text{ for all } x \in [0, 4\epsilon], \text{ where } C(\epsilon) = \max_{x\in[0,4\epsilon]} |F'(x) - \tilde{f}'(x)|. \tag{C.3}$$

Observe that the number of terms in the sum in the definition of $S^{(1)}$ is bounded by $1 + D_t^-(\epsilon) + U_t^+(\epsilon)$ where $D_t^-(\epsilon)$ and $U_t^+(\epsilon)$ denote the numbers of down-crossings of the band $(a-2\epsilon, a-\epsilon)$ and up-crossings of $(a+\epsilon, a+2\epsilon)$ by X before time t. Thus the expectation of $|S_t^{(1)}|$ can be bounded as follows:

$$\mathbb{E}_x[|S_t^{(1)}|] \leq 8\epsilon\, \mathbb{E}_x[1 + D_t^-(\epsilon) + U^+(\epsilon)]\, C(\epsilon). \tag{C.4}$$

Since X is a sub-martingale, the up-crossing lemma implies that the expected number of up-crossings $U_t^+(\epsilon)$ of the band $(c,d) = (a+\epsilon, a+2\epsilon)$ by time t does not grow faster than ϵ^{-1}: $\epsilon \cdot \mathbb{E}_x[U_t^+(\epsilon)] \leq \mathbb{E}_x[(X_t - d)^+] - \mathbb{E}_x[(X_0 - c)^+]$. Thus, it follows that $\epsilon \cdot \mathbb{E}_x[U_t^+(\epsilon)]$ remains bounded as $\epsilon \to 0$. As the number of down-crossings $D_t^-(\epsilon)$ is bounded by two added to the number of up-crossings $U_t^-(\epsilon)$ of the band $(a-2\epsilon, a-\epsilon)$, $\epsilon \cdot \mathbb{E}_x[D_t^-(\epsilon)]$ also remains bounded. Since $C(\epsilon)$ tends to zero as $\epsilon \to 0$, on account the facts that F and $\tilde{f}$ are $C^1(\mathbb{R}_+)$ and $F'(0) = \tilde{f}'(0)$ (cf. Eqn. (A.7), recalling that X is assumed to have unbounded variation), it thus follows from Eqn. (C.4) that $\mathbb{E}_x[|S_t^{(1)}|]$ tends to 0 as ϵ tends to zero, and the proof is complete. □

C.3. **Proof of Lem. 5.13:** Consider the function $\overline{G} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\overline{G}(a) = \sup_{b_-,b_+\geq 0} G^{(a)}(b_-, b_+)$. The fact that $\alpha^* > 0$ is a consequence of the intermediate value theorem and the following three assertions concerning $\overline{G}$:

(a) $\overline{G}(0) < 0$,
(b) There exists an $a_0 > 0$ such that $\overline{G}(a_0) > 0$ and
(c) The function $a \mapsto \overline{G}(a)$ is continuous at $a \in [0, a_0]$.

Assertion (a) follows directly from As. 1(iii). To verify assertion (b) we will show that for some a, b_-, b_+ with $a < b_+$, $G^{(a)}(b_-, b_+)$ is strictly positive. In view of the form of $G^{(a)}$ we thus need to show the existence of a triplet a, b_-, b_+, such that $F^{(a)\prime}(b_+ - a) > 1$ when $K = 0$, and such that $F^{(a)}(b_+ - a) - F^{(a)}(b_- - a) > b_+ - b_-$ when $K > 0$. As. 1(i, ii) and the right-continuity of the map

$J : (0,\infty) \to \mathbb{R}$ defined as follows:

$$J(y) := {}_0\mathcal{L}^w_\infty f(y) = \psi'(0) - q(y + w(0)) + \int_{(y,\infty)} [w(y-z) - w(0) + z - y]\nu(\mathrm{d}z),$$

imply that the following statement holds true:

(C.5) There exists an interval $I = [u_-, u_+]$, with $0 < u_- < u_+$, such that $J(y) > 0$ for all $y \in I$.

The assertion (b) is a direct consequence of the observation in Eqn. (C.5), the form of $G^{(a)}$ and the fact that, for any $a > 0$, $F^{(a)\prime}$ can be explicitly expressed in terms of J by $F^{(a)\prime}(x) = 1 - \int_0^x J(a + x - z)W^{(q)}(\mathrm{d}z)$. The last statement in turn follows on account of As. 1(i) and the definition of $F^{(a)}$ (see Appendix A.1(v), Eqn. (A.6)).

The finiteness of $\beta^*_+(\alpha^*)$ follows by an analogous reasoning as in Rem. 5.2. If either $K > 0$ or $K = 0$ and X has unbounded variation, then the equality $\alpha^* = \beta^*_+(\alpha^*)$ would imply that $V_{\alpha^*,\beta^*} \equiv f_w$—however, under As. 1(ii), there exist α, β such that $V_{\alpha,\beta}(x) > f_w(x)$ for $x \in (\alpha, \beta)$ (cf. Rem. 5.6), which yields a contradiction. □

Appendix D. Proof of optimality of single dividend-band strategies

The following auxiliary result provides a key-step in the proof:

Lemma D.1. *For $\theta > \Phi(q)$, the Laplace transform $g^*(\theta) := \int_0^\infty \mathrm{e}^{-\theta x} g(x)\mathrm{d}x$ of the function*

(D.1) $$g : (0,\infty) \to \mathbb{R} \qquad x \mapsto g(x) := {}_{b_+}(\mathcal{L}^w v_b - q\, v_b)(x)$$

is equal to $-\Xi(\theta)$ where

$$\Xi(\theta) = -\frac{\mathrm{e}^{\theta b_+}}{\theta} \int_{(b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G_{b_-}(\mathrm{d}z),$$

where $G_{b_-}(x) := G(b_-, x)$. In particular, the function $\theta \mapsto \Xi(\theta + \Phi(q))$ is completely monotone if and only if g is non-positive.

Proof of Thm. 5.4(ii): The assertion directly follows on account of Lem. D.1 and the fact that a necessary and sufficient condition for optimality of the policy π_{b^*} is that the function g given in Eqn. (D.1) satisfies $g(x) \le 0$ for all $x > b^+_*$ (Rem. 5.6(ii)). □

Thm. 5.4(iii) follows by combining Thm. 5.4(ii) with the following observation:

Lemma D.2. *If $x \mapsto G^*(x)$ is decreasing on (b^*_+, ∞), then $\Xi(\theta)$ is completely monotone on $(\Phi(q), \infty)$.*

Lem. D.2 will be proved in the next section.

D.1. **Key representation.** The following result provides an explicit connection between the function G and the infinitesimal generator:

Proposition D.3. *Let $c > 0$ and $b_+ \geq b_- \geq 0$ (with $b_+ \neq b_-$ if $K > 0$). (i) Then the following identity holds true:*

$$W^{(q)\prime}(b_+ + c)[G(b_-, b_+ + c) - G(b_-, b_+)] = \int_{[0,c]} ({}_{b_+}\mathcal{L}_\infty^{v_b} v_b)(b_+ + c - y)W^{(q)}(\mathrm{d}y) = 1 - {}_{b_+}F'_{v_b}(c). \tag{D.2}$$

(ii) If $G(b_-, b_+ + c) \leq G(b_-, b_+)$, then ${}_{b_+}F'_{v_b}(c) \geq 1$.

(iii) The functions $y \mapsto G(b^-, y)$ and $y \mapsto G^{\#}(y)$ are decreasing for all y sufficiently large.

The proof of Prop. D.3 is based on the following representation which is itself a consequence of the Shifting Lemma and the Pasting Lemma:

Lemma D.4. *For any $c > 0$ and any $b_+ \geq b_- \geq 0$, (with $b_+ \neq b_-$ if $K > 0$) the following identity holds true for any $x \leq b_+ + c$:*

$$\mathbb{E}_x\left[\mathrm{e}^{-q(t\wedge\tau_{b+c})}v_b(U^{b+c}_{t\wedge\tau_{b+c}}) + \int_0^{t\wedge\tau_{b+c}} \mathrm{e}^{-qs}\mathrm{d}D_s^{b+c}\right] - v_b(x) = \mathbb{E}_x\left[\int_0^{t\wedge\tau_{b+c}} \mathrm{e}^{-qs}({}_{b_+}\mathcal{L}_\infty^{\overline{w}}v_b)(U^{b+c}_{s-})\mathbf{1}_{\{U^{b+c}_{s-}>b_+\}}\mathrm{d}s\right]. \tag{D.3}$$

where $\overline{w} = v_{v_b}$ and we denoted $\tau_{b+c} = \tau^{\pi(b_-,b_++c)}$, $D^{b+c} = D^{\pi(b_-,b_++c)}$ and $U^{b+c} = U^{\pi(b_-,b_++c)}$.

Proof of Prop. D.3: First consider the case $K = 0$. Denoting the q-resolvent of Y^{b_++c} killed upon entering $(-\infty, 0)$ by

$$R^q_{0,b_++c}(x, \mathrm{d}y) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(Y_t^{b_++c} \in \mathrm{d}y, t < \tau_0)\mathrm{d}t$$

and letting $t \to \infty$ in (D.3) the dominated convergence theorem implies that for $x \in (0, b_+ + c)$

$$v_{b+c}(x) - v_b(x) = \mathbb{E}_x\left[\int_0^{\tau_{b+c}} \mathrm{e}^{-qs}[{}_{b_+}\mathcal{L}_\infty^{\overline{w}}v_b](U^{b+c}_{s-})\mathbf{1}_{\{U^{b+c}_{s-}>b_+\}}\mathrm{d}s\right] = \int_{[b_+,b_++c]} [{}_{b_+}\mathcal{L}_\infty^{\overline{w}}v_b](y)R^q_{0,b_++c}(x, \mathrm{d}y),$$

where $\overline{w} = v_b$. Inserting the explicit expressions (5.2) and (2.28) for v_b, v_{b+c} and $R^q_{0,b_++c}(x, \mathrm{d}y)$, we find that

$$W^{(q)}(x)[G^*(b_+ + c) - G^*(b_+)] = W^{(q)}(x)\int_{[b_+,b_++c]} [{}_{b_+}\mathcal{L}_\infty^{\overline{w}}v_b](y)\frac{W^{(q)}(b_+ + c - \mathrm{d}y)}{W^{(q)\prime}(b_+ + c)}, \qquad x \in (0, b_+ + c),$$

where we used that $W^{(q)}(x) = 0$ for $x < 0$. Changing coordinates in the integral and using that $W^{(q)}(x) > 0$ for $x > 0$ yields the first equality in Eqn. (D.2). The second equality in Eqn. (D.2) follows, for any $b_+ > 0$, by the representation in Eqn. (A.6). The case $b_+ = 0$ follows by approximation, taking

the limit of b_+ to zero. The proof of the case $K > 0$ is similar and omitted. The statement in (ii) is a direct consequence of Eqn. (D.2). The ultimate monotonicity of $y \mapsto G(b_-, y)$ and $y \mapsto G^{\#}(y)$ follows from the fact that ${}_{b_+}\mathcal{L}^w_\infty v_b(x)$ tends to minus infinity when $x \to \infty$. $\square$

Proof of Lem. D.1: Taking the Laplace transform in c in Eqn. (D.2) and using the form of the Laplace transform of $W^{(q)}$ yields that, for $\theta > \Phi(q)$,

$$\begin{aligned} g^*(\theta) \cdot \frac{\theta}{\psi(\theta) - q} &= \int_{[0,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(b_+ + c)[G^*(b_+ + c) - G^*(b_+)]\mathrm{d}c \\ &= \int_{[0,\infty)} \int_{[z,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(b_+ + c)\mathrm{d}c\, G^*(b_+ + \mathrm{d}z) \\ &= \mathrm{e}^{\theta b_+} \int_{[b_+,\infty)} \int_{[z,\infty)} \mathrm{e}^{-\theta c} W^{(q)}(\mathrm{d}c) G^*(\mathrm{d}z) = \frac{\mathrm{e}^{\theta b_+}}{\psi(\theta) - q} \int_{[b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,v)\prime}(z) G^*(\mathrm{d}z) \end{aligned}$$

by a change of the order of integration, justified by Fubini's theorem, and the form (A.11) of $Z^{(q,v)\prime}(z)$. Comparison with Ξ defined in (D.2) shows that $g^*(\theta) = -\Xi(\theta)$ for $\theta > \Phi(q)$. The final assertion follows since a function $f : (0,\infty) \to \mathbb{R}$ is completely monotone if and only if it is the Laplace transform of a measure. $\square$

Proof of Lem. D.2: 1. If $f(\theta)$ is the Laplace transform of the measure μ on $\mathbb{R}_+$ then, for any $c > 0$, $\theta^{-1}\mathrm{e}^{-\theta c} f(\theta)$ is the Laplace transform of the function $y \mapsto \mathbf{1}_{\{y-c \in \mathbb{R}_+\}} \mu([0, y-c])$. Since $Z^{(q,\theta)\prime}(x)$ is completely monotone for any $x > 0$ (cf. Rem. A.4(ii)) it follows that for any $x, b > 0$, the function

$$\theta \mapsto \theta^{-1} \mathrm{e}^{-\theta(x-b)} Z^{(q,\theta)\prime}(x)$$

is completely monotone on $(\Phi(q), \infty)$.

2. If $\theta \mapsto f_x(\theta)$, $x > b > 0$, is a collection of completely monotone functions and μ is a measure on (b,∞) then it is straightforward to verify that $\theta \mapsto \int_{(b,\infty)} f_x(\theta)\mu(\mathrm{d}x)$ is also completely monotone. Hence, if G^* is decreasing, complete monotonicity of Ξ follows on account of the complete monotonicity of $\theta^{-1}\mathrm{e}^{\theta(b-x)} Z^{(q,\theta)\prime}(x)$ and the form of Ξ given in Eqn. (D.2). $\square$

D.2. **Proof of Thm. 5.7.** In view of Rem. 5.6, it suffices to verify that Eqn. (5.12) is satisfied under conditions (a) or (b).

Proof of (5.12) *under condition (a):* We need to show that $J(x) \le 0$ for all $x > 0$ where $J : (0,\infty) \to \mathbb{R}$ is given by $J(x) := ({}_{b_+^*}\mathcal{L}^{\tilde{w}}_\infty v_{b^*})(b_+^* + x)$ with $\tilde{w} = v_{b^*}$. In view of the forms of the operator ${}_{b_+^*}\mathcal{L}^w_\infty$ and of $v_{b_+^*}(x)$ for $x > b_+^*$, it follows that $J(x)$ is given by the following expression:

$$J(x) = \psi'(0) - q(x + v(b)) + \int_0^\infty [v(b-y) - v(b) + y]\nu'(x+y)\mathrm{d}y, \qquad x > 0, \tag{D.4}$$

where we denoted $b = b_+^*$ and $v = v_{b^*}$.

The assertion that $J(x) \le 0$ for any $x > 0$ then follows once we show that (i) J is concave on $(0,\infty)$ (ii) $J(0+) = 0$ and (iii) $J'(0+) \le 0$.

To show (i) note that, under condition (a), the integrand in (D.4) is non-positive for all y. Indeed, for $y \in (0,b)$, $[v(b-y)-v(b)+y] \leq 0 \Leftrightarrow v(b)-v(b-y) \geq y$ (as $K=0$), and for $y \geq b$ we have that $w(b-y)-v(0)-b+y \leq 0$ and $v(0)-v(b)+b \leq 0$ which yields that $w(b-y)-v(b) \leq y$ for $y \geq b$. As ν' is convex, and a mixture of convex functions with positive weights is again convex, we deduce that J is concave on $(0,\infty)$.

Given (ii) statement (iii) follows since if $J'(0+)$ were positive, $(J(x)-J(0))/x = J(x)/x$ would be positive which would be in contradiction with Eqn. (D.5).

To see that (ii) holds, note that, from (D.2) with $b_- = b_-^*$ and $b_+ = b_+^*$,

$$0 \geq \int_{[0,c]} J(c-y) W^{(q)}(\mathrm{d}y) \qquad \text{for all } c > 0. \tag{D.5}$$

Thus, we deduce that $J(0+) \leq 0$.

To complete the proof we next verify that $J(0+) = 0$. First consider the case that σ^2 is strictly positive: The observations that, for any $b > 0$, $\mathrm{e}^{-q(t\wedge T_{0,b})} v_b(X_{t\wedge T_{0,b}})$ is a martingale and $v_b \in C^2$ together with Itô's lemma yield that $({}_0\mathcal{L}^w_\infty v_b)(x) = 0$ for all $x \in (0,b_+)$ which in turn implies that $J(0) = ({}_0\mathcal{L}^w_\infty v_b)(b_+) = 0$ on account of the continuity of $x \mapsto ({}_0\mathcal{L}^w_\infty v_{b^*}(x)$ at $x = 0$.

Consider next the case $\sigma^2 = 0$ and $\int_0^1 x\nu(\mathrm{d}x) < \infty$. It follows by taking Laplace transforms in Eqn. (4.6) that $({}_0\mathcal{L}^w_\infty v_b)(x) = 0$ for Leb-a.e. $x \in (0,b_+)$. Let $x_n \in (0,b_+^*)$ be a sequence tending to b satisfying $({}_0\mathcal{L}^w_\infty v_b)(x_n) = 0$. On account of Fatou's lemma, the convexity of ν', the continuity of $W^{(q)}|_{(0,\infty)}$ and $W^{(q)\prime}|_{(0,\infty)}$ and the fact that $v(b-y)-v(b) \leq y$ for all $y \leq b$, we deduce that $J(0+) \geq 0$:

$$0 = \lim_n ({}_0\mathcal{L}^w_\infty v_b)(x_n) \leq \psi'(0) - qv(b) + \int_0^\infty v(b-y)-v(b)+y)\nu'(y)\mathrm{d}y = J(0+).$$

Hence also in the case that X has bounded variation it holds that $J(0+) = 0$.

The case $\sigma^2 = 0$ and $\int_0^1 x\nu(\mathrm{d}x) = \infty$ follows by approximation: by adding a small Brownian component with variance $\sigma^2 > 0$ to X and then letting $\sigma^2 \to 0$, it follows that also in this case $J(0+) = 0$.

To verify this claim we show that $J(0+) \geq 0$. If $\sigma \searrow 0$, the continuity theorem implies that the scale functions $W^{(q)(\sigma)}$ and $F_w^{(\sigma)}$ of $X + \sigma^2 W$ converge pointwise to the corresponding scale functions of X at any point of continuity. Denote by $J^{(\sigma)}(x)$ the expression on the rhs of (D.4) with the function v replaced by the function $v^{(\sigma)}$ that is equal to v except for $x \in [0,b]$ where it is given by $v^{(\sigma)}(x) = W^{(q)(\sigma)}(x)G + F_w^{(\sigma)}$ with $G = G^\#(b)$ (independent of σ). An application of Fatou's lemma, which is justified on account of the bounds in Lem. 4.8 and Lem. 4.9, then yields that

$$0 = \lim_{\sigma\searrow 0} J^{(\sigma)}(x) \leq J(x), \qquad \text{for any } x > 0.$$

The proof is complete. □

Proof of under condition (b) Since ν' is completely monotone, there exists a measure ξ on $(0,\infty)$ such that $\nu'(y) = \int_0^\infty \mathrm{e}^{-\mu y}\xi(\mathrm{d}\mu)$. Hence, by interchanging the order of integration, justified by Fubini's

theorem, we find that

$$J(x) = \psi'(0) - q(x + v(b)) + \int_0^\infty \mathrm{e}^{-\mu x} B(\mu)\xi(\mathrm{d}\mu),$$

where $B(\mu) = \int_0^\infty \mathrm{e}^{-\mu y}[v(b-y) - v(b) + y]\nu(\mathrm{d}y)$. In this case J is continuously differentiable on $(0,\infty)$ with derivative

$$J'(x) = -q - \int_0^\infty \mathrm{e}^{-\mu x}\mu B(\mu)\xi(\mathrm{d}\mu).$$

By a similar reasoning as in part (a), it follows from Eqn. (D.5) that $J'(0+) \in [-\infty, 0)$, which yields that $-\int_0^\infty \mu B(\mu)\xi(\mathrm{d}\mu) \le q$. In particular, there exists some function $C : \mathbb{R}_+ \to \mathbb{R}_+$ with $C(\mu) \ge -\mu B(\mu)$ such that $q = \int_0^\infty C(\mu)\xi(\mathrm{d}\mu)$. Hence, we have for any $x > 0$:

$$\begin{aligned} J'(x) &= \int_0^\infty [-\mu B(\mu)\mathrm{e}^{-\mu x} - C(\mu)]\xi(\mathrm{d}\mu) \\ &= \int_0^\infty [-\mu B(\mu) - C(\mu)]\mathrm{e}^{-\mu x}\xi(\mathrm{d}\mu) - \int_0^\infty C(\mu)[1 - \mathrm{e}^{-\mu x}]\xi(\mathrm{d}\mu) \le 0, \end{aligned}$$

in view of the definition of C. Since $J(0+) = 0$ it thus follows that $J(x) = 0$ for all $x > 0$, which establishes (5.12) under condition (b). The proof of Theorem 5.7 is complete. □

References

[1] Albrecher, H. and Thonhauser, S. (2007) Dividend maximization under consideration of the time value of ruin, *Insurance, Mathematics and Economics* **41**, pp. 163–184.

[2] Albrecher, H., and Thonhauser, S. (2008) Optimal dividend strategies for a risk process under force of interest, *Insurance, Mathematics and Economics*, **43**, pp. 134–149.

[3] Albrecher, H., and S. Thonhauser (2009). Optimality results for dividend problems in insurance. *RACSAM Revista de la Real Academia de Ciencias, Serie A, Matemticas* **103**, pp. 295-320.

[4] Alvarez, L. H. R. and Virtanen, J. (2006) A class of solvable stochastic dividend optimization problems: on the general impact of flexibility on valuation. *Economic Theory* **28(2)**, pp. 373–398.

[5] Alvarez, L. H. R. and Rakkolainen, T. A. Optimal payout policy in presence of downside risk. *Math. Methods Oper. Res.*, 69(1):2758, 2009.

[6] Asmussen, S., Højgaard, B. and Taksar, M. (2000) Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation, *Finance Stoch.* **4**, pp. 299–324.

[7] Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004) Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options, *Ann. Appl. Probab.* **14**, pp. 215–238.

[8] Avram, F., Palmowski, Z. and Pistorius, M.R. (2007) On the optimal dividend problem for a spectrally negative Lévy process, *Ann. Appl. Probab.* **17**, pp. 156–180.

[9] Avram, F., Palmowski, Z. and Pistorius, M.R. (2010) On optimal dividend distribution for a Cramer-Lundberg process with exponential jumps in the presence of a linear Gerber-Shiu penalty function. *Monografías del Seminario Matemático García de Galdeano* (to appear).

[10] Azcue, P. and Muler, N. (2005) Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, *Mathematical Finance* **15**, pp. 261–308.

[11] Bayraktar, E. and Sîrbu, M. (2011) Stochastic Perron's method and verification without smoothness using viscosity comparison: the linear case. *Proc. Am. Math. Soc.* (to appear).

[12] Bertoin, J. (1996) *Lévy processes*, Cambridge University Press.

[13] Bertoin, J. (1997) Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval, *Ann. Appl. Probab.* **7**, pp. 156–169.

[14] Biffis, E. and Kyprianou, A.E. (2010) A note on scale functions and the time value of ruin for Lévy insurance risk processes. *Insurance Mathematics and Economics* 46, pp. 85–91.

[15] Bingham, N.H. (1975) Fluctuation theory in continuous time, *Adv. Appl. Probab.* **7**, pp. 705–766.

[16] Dickson, D.C.M. and Waters, H.R. (2004) Some optimal dividends problems, *ASTIN Bulletin* **34**, pp. 49–74

[17] De Finetti, B. (1957) Su un'impostazione alternativa dell teoria colletiva del rischio, *Trans. XV Intern. Congress Act.* **2**, pp. 433–443.

[18] Dynkin, E. B. (1963) The optimum choice for stopping a Markov process. *Soviet Math. Dokl.* **4**, pp. 627–629.

[19] Gerber, H.U. (1969) Entscheidungskriterien für den Zusammengesetzten Poisson Prozess. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* **69**, pp. 185-228.

[20] Gerber, H.U. (1972) Games of Economic Survival with Discrete- and Continuous-Income Processes. *Operations Research* **20**, pp. 37–45.

[21] Gerber, H.U. (1979) *An introduction to mathematical risk theory,* Hübner Foundation for Insurance Education series 8, Philadelphia.

[22] Gerber H. U. and Lin X. S. and Yang H (2006) A Note on the Dividends-Penalty Identity and the Optimal Dividend Barrier. *Astin Bulletin* **36(2)**, pp. 489-503.

[23] Gerber, H.U. and Shiu, E.S.W. (2004) Optimal dividends: analysis with Brownian motion, *North American Actuarial Journal* **8**, pp. 1–20.

[24] Gerber, H.U., Lin, X.S. and Yang, H. (2006) A note on the Dividends-Penalty Identity and the Optimal Dividend Barrier, *Astin Bull.* **36**, pp. 489-503.

[25] Grandits, P., Hubalek, F., Schachermayer, W., and Zigo, M. (2007) Optimal expected exponential utility of dividend payments in Brownian risk model, *Scandinavian Actuarial Journal* **2**, pp. 73–107.

[26] Hallin, M (1979) Band strategies: The random walk of reserves, *Blätter der DGVFM* **14**, pp. 321–236.

[27] Irbäck, J. (2003) Asymptotic theory for a risk process with a high dividend barrier, *Scand. Actuarial J.* **2**, pp. 97–118.

[28] Jeanblanc, M. and Shiryaev, A.N. (1995) Optimization of the flow of dividends, *Russian Math. Surveys* **50**, pp. 257–277.

[29] Kulenko, N. and Schmidli, H. (2008). Optimal Dividend Strategies in a Cramér-Lundberg Model with Capital Injections. *Insurance Math. Econom.* **43**, 270–278.

[30] Kyprianou, A.E. (2006) *Introductory lectures on fluctuations of Lévy processes with applications*, Springer.

[31] Kyprianou, A.E., Rivero, V. and Song, R. (2010) Convexity and smoothness of scale functions and de Finettis control problem, Journal of Theoretical Probability 23, pp. 547-564.

[32] Lambert, A. (2000) Completely asymmetric Lévy processes confined in a finite interval, *Ann. Inst. H. Poincaré Prob. Stat.* **36**, pp. 251–274.

[33] Loeffen, R. (2008) On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes, *Ann. Appl. Probab.* **18**, pp. 1669–1680.

[34] Loeffen, R. (2009) An optimal dividends problem with transaction costs for spectrally negative Lévy processes, *Insurance: Mathematics and Economics* **45**, pp. 41-48.

[35] Loeffen, R. and Renaud, J-F. (2009) De Finetti's optimal dividends problem with an affine penalty function at ruin. *Insurance: Mathematics and Economics* **46**, pp. 98-108.

[36] Neveu, J. (1975) *Discrete Parameter Martingales* North Holland, Amsterdam.

[37] Peskir, G. and Shiryaev A. (1978) *Optimal Stopping and Free-Boundary Problems*. Springer.

[38] Pistorius, M.R. (2004) On exit and ergodicity of the completely asymmetric Lévy process reflected at its infimum, *J. Th. Probab.* **17**, pp. 183–220.

[39] Pistorius, M.R. (2005) A potential-theoretical review of some exit problems of spectrally negative Lévy processes, *Sem. Probab.* **38**, pp. 30-41.

[40] Protter, P. (1995) *Stochastic Integration and differential equations*, Spinger Verlag.

[41] Sato, K. (1999) *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge.

[42] Schmidli, H. (2008) *Stochastic Control in Insurance*, Springer Verlag.

[43] Shreve, S. E., Lehoczky, J. P. and Gaver, D. P. (1994) Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control and Optimization* **22**, pp. 55-75.

[44] Stroock, D.W. and Varadhan, S.R.S. (1972) On degenerate elliptic-parabolic operators of second order and their associated diffusions. *Comm. Pure and Appl. Math.* **25**, pp. 651–713.

[45] Yuen, K.C., Wang, G. and Li, W.K. (2006) The Gerber-Shiu expected discounted penalty function for risk processes with interest and a constant dividend barrier, *Insurance: Mathematics and Economics* **40**, pp. 104–112.

[46] Zajic, T (2000) Optimal Dividend Payout under Compound Poisson Income, *J. Opt. Theor. Appl.* **104**, pp. 195–213.

[47] Zhou, X. (2005) On a classical risk model with a constant dividend barrier, *North American Actuarial Journal* **9**, pp. 1–14.

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